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Type IIB 2-form fields and gauge coupling constant of 4D $\mathcal{N} = 2$ super QCD

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Abstract. We study the relation between the Type IIB (NSNS and RR) 2-form fields and the (complex) gauge coupling constant of the 4D $\mathcal{N} = 2$ $SU(N_c)$ super Yang–Mills theory with N_f fundamental matter particles. We start from the analysis of the D2-brane world-volume theory with heavy N_c quarks on the N_f D6 supergravity background. After a sequence of T- and S-dualities, we obtain the (generalized) 2-forms in the configuration with N_c D5-branes wrapping on a vanishing two-cycle under the influence of the background. These 2-forms show the same behavior as the gauge coupling constant of the 4D $\mathcal{N} = 2$ super QCD. The background reduces to the Z_{N_f} orbifold in the twelve-dimensional space-time formally realized by introducing the two parameters as additional space coordinates. The 10D gravity dual is suggested as the 2D flip in this twelve-dimensional space-time. In the case of $N_f = 2N_c$, this gravity dual becomes $AdS_5 \times S^5/Z_2$ with a D3-charge which depends on the constant generalized NSNS 2-form. This is the result expected from the M-theory QCD configuration. Based on the known exact result, we also discuss this configuration after including non-perturbative effects.

1 Introduction and conclusions

Recently, the generalization of the AdS_5/CFT_4 correspondence [1] to the non-conformal cases has been actively studied. The first success is the discovery of the correspondence between the Type IIB (NSNS and RR) 2-form fields and the complex (including theta parameter) gauge coupling constant of the 4D non-conformal super Yang–Mills (SYM) theory realized on this configuration [2]. After this discovery, the corresponding supergravity (SUGRA) solution has also been constructed for $\mathcal{N} = 2$ [3] and $\mathcal{N} = 1$ [4] non-conformal SYM theories. Especially, the case with $\mathcal{N} = 2$ supersymmetry is easy to handle due to its higher supersymmetry. It is also useful to discuss the theories with $\mathcal{N} = 1$ supersymmetry or without supersymmetry. A lot of studies have been done in this direction [5–15].

Above all, the gauge theory with the fundamental matter particles (fundamentals) has attracted a lot of interest. One reason for this is that the ratio between the number of flavors and the rank of the gauge theory appears with the typical coefficient in the behavior of the one-loop renomalization group (RG) flow of the gauge coupling constant. We can compare this fact with the behavior of the corresponding SUGRA fields.[1] This will make manifest the relation between the radial coordinate of the SUGRA solution and the energy scale of the field theory, as originally suggested in [1].

Some attempts have been made [16, 17] to solve this problem in the 4D $\mathcal{N} = 2$ case. But, as found and discussed in [16], the obtained SUGRA solution does not show the properties required for the gravity dual corresponding to the 4D $\mathcal{N} = 2$ field theory.[2]

For example, if we can reproduce the correct RG-flow on the supergravity side, this RG-flow will vanish in the special case corresponding to the 4D $\mathcal{N} = 2$ CFT. In this case, we can also expect that the SUGRA solution will have the structure of AdS_5, which has the conformal symmetry. This is the required condition coming from the AdS/CFT correspondence and gives us a check whether the analysis is correct or not. But, as commented in [16], this approach does not satisfy this condition. So this problem remains unsolved and we need some modifications of this approach to obtain the correct result. This is the main purpose of this paper and the motivation of this study.

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[1] When we quantitatively examine AdS/CFT correspondence, we have to compare the ratio of the two different kinds of quantity to kill the normalization with the convention dependence. For example, the gauge coupling constant of 4D $\mathcal{N} = 4$ SYM on D3-branes is proportional to the string coupling constant g_s, but the coefficient is the convention dependent. In [19], this coefficient is fixed by requiring the ratio between the tension of the fundamental string and that of the D-string to be $1/g_s$

[2] In [17], an interpretation was suggested to the effect that the whole string perturbation effect is included in this SUGRA solution

On the other hand, there is the well-known successful example of how to make clear the structure of the 4D field theory vacua in Type IIA string theory – M-theory QCD (MQCD) [18]. In this model, the analysis of supersymmetric cycle enables us to study 4D gauge field theory including non-perturbative effect. To solve the above difficult problem, it would be the best way to reconsider this successful model. By getting to know the situation and how this model works, we may find the solution to the above mentioned problem.[3] In fact, it is suggested that the Type IIB configurations corresponding to the SUGRA solutions are the T-dual of the MQCD configurations [20,21]. In [22], it is explicitly confirmed that a special kind of Type IIB SUGRA solution such as $\mathrm{AdS}_5 \times \mathrm{S}^5/\mathrm{Z}_2$ [23] is the T-dual of the Type IIA SUGRA solution corresponding to the MQCD configuration.

There is the well-known MQCD configuration corresponding to the 4D $\mathcal{N}=2$ $SU(N_c)$ super Yang–Mills theory with N_f quarks in the fundamental representation. The above equivalence by the T-duality means that there is also the Type IIB configuration corresponding to this field theory. Therefore we can expect that we will have the Type IIB SUGRA dual from this configuration. This SUGRA dual will reproduce the typical RG-flow of this field theory and have the AdS_5 structure for the special value $N_f = 2N_c$. For this purpose, we need a careful treatment of this T-duality in order to apply our knowledge of the MQCD analysis. By this analysis, it will be possible to find how to modify the previous approach to obtain the correct result.

In this paper, we follow this direction and study the relation between the Type IIB 2-forms and the (complex) gauge coupling constant of the 4D $\mathcal{N}=2$ $SU(N_c)$ super Yang–Mills theory with N_f quarks in the fundamental representation. The outline of the strategy and results of this paper is as follows.

First, we reconsider the MQCD configuration with the N_f D6 background. We can find that the RG-flow of the gauge coupling constant is determined not by the simple difference of the position coordinates for the two NS5-branes, but by that of the newly defined coordinate. From this observation we can point out that it will be also true for the (T-dualized) Type IIB system; a newly defined field (twisted sector) is required for the gauge coupling constant. The use of a different type of twisted sector is one of the points most different from the previous attempts [16,17]. This gives us a clue about how to treat the problem.

For this purpose, we start from the analysis of one D2-brane world-volume theory with heavy N_c quarks on the N_f D6 supergravity background. This is the T- and S-dual of the MQCD-like configuration; N_c semi-infinite D4-branes terminating on one NS5-brane under a N_f D6 supergravity background. This D2-brane model is easier to handle, so we study this configuration. In fact, at this stage we can see that the behavior of the newly defined world-volume field is the same as that of the RG-flow of the gauge theory. We can simply generalize this analysis to the case with two D2-branes.

Here, as a by-product of this analysis, we can also show that there are two kinds of definitions for the electric charge on the D2-branes according to their relative positions to the D6-branes. This relative difference will be related to the string creation (the so-called Hanany–Witten effect [24]), but this shows that the observer on the D2-brane does not see the string creation. We can also show that there are $N_f + 1$ inequivalent BPS configurations which cannot be continuously transformed into each other. This corresponds to the s-rule[4] [24].

Next, we estimate the form of the background on the D3-brane which is the T-dual of the previous D2-brane. Here we have to emphasize that we use the correspondence between the world-volume scalar and the Wilson line under the T-duality. As a result of that, the background is different from the D7 SUGRA solution, although the background is the D6 SUGRA solution in Type IIA. This is the another point most different from the previous attempts [16,17]. This analysis corresponds to treating this configuration as the D3-brane on the background of a D7 SUGRA solution. But remember that one special limit for the compactified radius is required to get the D7 SUGRA solution from the D6 SUGRA solution. We have to check whether this limit is consistent with the condition for the realization of the field theory. Then we can see that it is only by our procedure that we can transfer the successful result of the Type IIA configuration to that of the Type IIB configuration.

Next we rewrite the world-volume gauge field in terms of the NSNS and RR 2-forms in Type IIB theory. After a sequence of T- and S-dualities we obtain the generalized Type IIB 2-forms in the configuration we aimed to; N_c D5-branes wrapping on the vanishing two-cycle between the two Kaluza–Klein (KK) monopoles under the influence of the background. These 2-forms show the same logarithmic behavior as the RG-flow of the 4D $\mathcal{N}=2$ $\mathrm{SU}(N_c)$ Super QCD (SQCD) with N_f flavors.

Here we have to note that these 2-form fields originate from the world-volume fields corresponding to the two-dimensional space in M-theory. This two-dimensional space is related by T-duality to the torus with the complex structure made up of a dilaton and an axion in Type IIB theory [26]. In other words, the above 2-form fields correspond to the new coordinates of the additional two-dimensional space for Type IIB theory. By including these two degrees of freedom as the new space coordinates, we formally extend our discussion to twelve-dimensional space-time. Then we see that the background reduces to the Z_{N_f} orbifold in this twelve-dimensional space-time. Therefore our configuration reduces to the one embedded in this locally flat background. By this procedure, we can extract or separate the gravity induced by branes with (open string) dynamics from the background.

The other important point here is that we can separate the other two-dimensional space from this twelve-dimen-

[3] This line is also referred to in [16] as a future problem

[4] When this work was completed, we received the paper of [25], in which the authors have confirmed one aspect of the s-rule: that there is a maximum of N for the continuous string charge connected between N D3-branes and one D5-brane

sional space-time. This two-dimensional space decouples from the remaining ten-dimensional space-time. That is, what we have done is to add the extra two-dimensional space and to pick up the unimportant other two-dimensional space from the twelve-dimensional space-time. This is a procedure similar to the M-theory flip. On the other hand, in F-theory it is known that the extra two-dimensional space corresponds to the space for the dilaton and axion of Type IIB theory. In the context of F-theory, our procedure is replacing the two-dimensional space for a non-trivial dilaton and axion with another two-dimensional space for constant dilaton and axion. That is, we take the frame of the (remaining) ten-dimensional space-time in which the dilaton and axion are constant. In this frame, the generalized 2-form (twisted sector) becomes the ordinary one. Our suggestion is that this remaining ten-dimensional space-time would be the gravity dual of the corresponding field theory. Then we can find that the configuration in the remaining ten-dimensional space-time is qualitatively the same as that of pure SYM theory. This is consistent with the fact that at one-loop level, the structure of pure SYM vacua is qualitatively the same as the Coulomb branch of SQCD.

By applying the 10D SUGRA solution for pure SYM theory [3], we can obtain the explicit form of the aimed 10D gravity dual realized as the 2D flip in twelve dimensions. Especially, in the case of $N_f = 2N_c$, this 10D gravity dual reduces to $\mathrm{AdS}_5 \times \mathrm{S}^5/\mathrm{Z}_2$ with a D3-charge which depends on the constant generalized NSNS 2-form. This is the result expected from the corresponding MQCD configuration in which the N_c D4-branes are wrapping on only a part of the circle.

Until this stage, we study our configuration in the region where we can ignore the non-perturbative effect in 4D field theory. Based on the exact result known purely in the field theory, we speculate on how our configuration will be described. We find that the classical δ function-like singularities as the source of the D5-charges change into those of the branch cuts, and there is a new type of "flux"[5] which goes round between one branch cut and another branch cut.

This paper is organized as follows. In Sect. 2, we reconsider why a MQCD analysis for the $\mathrm{SU}(N_c)$ SQCD with N_f fundamental representation matter works well and anticipate what we should do in our configuration. In Sect. 3, we analyze the D2-brane world-volume field theory with heavy N_c quarks on the N_f D6-brane background. We show that in this stage, the behavior of the scalar field is the same as the RG-flow of the SQCD. In Sect. 4, we discuss how the previous result is transferred to the T-dualized configuration. In Sect. 5, we transform the field on the world-volume to those of the NSNS and RR 2-forms of Type IIB. In Sect. 6, by using the sequences of T- and S-dualities, we bring all our results to the aimed configuration. Then we discuss the correspondence between the Type IIB 2-forms and the RG-flow of the gauge theory. We also discuss the 10D gravity dual in the formal twelve dimensions case. In Sect. 7, based on the exact result, we speculate how our configuration is described when non-perturbative effects in 4D field theory are included.

[5] The quotation marks are added because the flux is originally the name for the RR and NSNS 2-form before taking the non-perturbative effect

2 Analysis of MQCD configuration revisited

Let us start from the analysis of the MQCD configuration [18]. This configuration consists of two NS5-branes and N_c D4-branes suspended between them on the N_f D6-branes background. These N_f D6-branes are also located between the two NS5-branes with respect to x^6-direction and we consider their positions as the origin in the directions of x^4, x^5 and x^6. We set their world-volume and locations as follows:

$$\begin{aligned}
\text{Two NS5}: &\quad 1\ 2\ 3\ 4\ 5\ -\ -\ -\ - \\
&\quad \text{at } x^6 = x^{6\pm},\quad x^7 = x^8 = x^9 = 0, \\
N_c\ \ D4: &\quad 1\ 2\ 3\ -\ -\ 6\ -\ -\ - \\
&\quad \text{at } x^4 = x^5 = x^7 = x^8 = x^9 = 0, \\
N_f\ \ D6: &\quad 1\ 2\ 3\ -\ -\ -\ 7\ 8\ 9 \\
&\quad \text{at } x^4 = x^5 = x^6 = 0.
\end{aligned}$$

Let us consider the N_f D6-branes as the background of the D6 SUGRA solution. The D6 solution is given by

$$\begin{aligned}
\mathrm{d}s_{10}^2 &= H^{-\frac{1}{2}}\left(\eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + \sum_{i=7,8,9}(\mathrm{d}x^i)^2\right) \\
&\quad + H^{\frac{1}{2}}(\mathrm{d}x_6^2 + |\mathrm{d}z|^2), \\
\mathrm{e}^\phi &= g_\mathrm{s} H^{-\frac{3}{4}}, \qquad G^R_{ab} = -\partial_c H \epsilon_{ab}{}^c, \qquad (1)\\
H &\equiv 1 + \frac{g_\mathrm{s}\alpha'^{\frac{1}{2}} N_f}{2r}, \qquad r \equiv \sqrt{|z|^2 + x_6^2}, \\
z &\equiv x^4 + \mathrm{i}x^5,
\end{aligned}$$

where e^ϕ and G^R_{ab} are the dilaton and the RR 2-form field strength. ϵ^{abc} is the 3-cyclic epsilon tensor with values $\epsilon^{456} = -\epsilon^{546} = \epsilon^{564} = +1$ and $\epsilon_{ab}{}^c = \epsilon_{abc}$ etc., and we denote the indices μ, ν and the metric $\eta_{\mu\nu}$ as the indices and the flat metric of the four-dimensional space-time x^0, x^1, x^2, x^3.

Let us consider the four-dimensional gauge theory on the N_c D4-branes. This gauge theory is realized on the D4-branes after the dimensional reduction in the direction of x_6. The renomalization group flow (RG-flow) of the gauge coupling constant is given by the equation

$$\begin{aligned}
\frac{1}{g_{\mathrm{YM}}^2} &= \frac{1}{\alpha'^{\frac{1}{2}}}\int_{x^-}^{x^+} \mathrm{d}x_6 \mathrm{e}^{-\phi}\sqrt{-g_{66}\det\{g_{\mu\nu}\}}\ H \\
&= \frac{1}{g_\mathrm{s}\alpha'^{\frac{1}{2}}}\int_{x^-}^{x^+} \mathrm{d}x_6 \left(1 + \frac{g_\mathrm{s}\alpha'^{\frac{1}{2}} N_f}{2\sqrt{|z|^2 + x_6^2}}\right)
\end{aligned}$$

$$= \frac{\Delta x_6}{g_s \alpha'^{\frac{1}{2}}} + \frac{N_f}{2} \ln\left(\frac{x_6^+ + \sqrt{|z|^2 + x_6^{2+}}}{x_6^- + \sqrt{|z|^2 + x_6^{2-}}}\right), \qquad (2)$$

where the additional factor H of the second equation is needed in order that the field strength with the up-indices $F^{\mu\nu}$ is defined as $F^{\mu\nu} = \eta^{\mu\rho}\eta^{\nu\lambda}F_{\rho\lambda}$.

Let us consider the limit in which the quantities appearing in the four-dimensional field theory will remain finite. We take the limit below[6]:

$$g_s \to 0, \quad \alpha' \to 0, \quad x^6 \to 0, \quad z \to 0;$$
$$\Lambda \equiv \frac{g_s}{4\pi\alpha'^{\frac{1}{2}}}, \quad \phi \equiv \frac{x^6}{2\pi\alpha'\Lambda}, \quad u \equiv \frac{z}{2\pi\alpha'} \text{ fixed.} \qquad (3)$$

Note that in this limit the above gauge coupling constant remains finite. Let us compare the gauge coupling constant of pure SYM with that of SQCD. In the case of pure SYM, the RG-flow of the gauge coupling constant is expressed by the distance Δx^6. But the different point in SQCD is the existence of the second term of (2). Due to this term, the gauge coupling constant is not expressed only by the distance Δx^6 and comes to depend on the explicit coordinates ($x^6_\pm$). It seems that we need two degrees of freedom to express the one degree of freedom for the gauge coupling constant. In this sense, the original coordinate x^6 is not as good as in the case of pure SYM theory. This suggests that we should take another coordinate by which we can treat the gauge coupling constant in the same way as pure SYM theory. Let us take the new coordinate which satisfies this requirement as

$$d\tilde{x}_6 \equiv dx_6 e^{-\phi}\sqrt{-g_{66}\det\{g_{\mu\nu}\}}\; H \quad \text{for fixed } z\ . \qquad (4)$$

By the integral with respect to x^6, we get

$$\frac{\tilde{x}_6}{g_s\alpha'^{\frac{1}{2}}} = \frac{x_6}{g_s\alpha'^{\frac{1}{2}}} + \frac{N_f}{2}\ln\left(\frac{x_6 + \sqrt{|z|^2 + x_6^2}}{2\pi\alpha'\Lambda}\right)$$
$$= \frac{\phi}{2} + \frac{N_f}{2}\ln\left(\frac{\phi\Lambda + \sqrt{|u|^2 + (\phi\Lambda)^2}}{\Lambda}\right), \qquad (5)$$

where we add the appropriate integral constant to make it dimensionless. By using this new coordinate, we get the behavior of the gauge coupling constant: $1/g^2_{\rm YM} = \Delta\tilde{x}^6/\left(g_s\alpha'^{\frac{1}{2}}\right)$. As a result of that, we can embed the whole effect of the D6-branes in the difference of the new coordinate $\Delta\tilde{x}^6$. This enables us to treat this field theory in the same way as the case on the flat background.

Next, let us consider the theta parameter of the field theory. The theta parameter is determined by the distance between the two NS5-branes in the direction of x^{10}. The Type IIA configuration is delocalized in this direction, but we can keep the distance between them as the phase. So the information about this relative distance remains even in Type IIA theory.[7] As a result of that, we obtain the relation between the theta parameter and the distance of the x^{10}-direction, as $\theta = \Delta x^{10}$, as suggested by [18]. For later convenience, we define another new coordinate χ as $\chi \equiv 2x^{10}/\left(g_s\alpha'^{\frac{1}{2}}\right)$.

Putting together $\tilde{x}_6$ and x_{10}, we define a new complex coordinate y as

$$\ln y \equiv \frac{\tilde{x}^6 + i x^{10}}{g_s\alpha'^{\frac{1}{2}}} = \frac{\tilde{\phi} + i\chi}{2} \qquad (6)$$
$$= \frac{1}{2}\left\{\phi + N_f \ln\left(\frac{\phi\Lambda + \sqrt{|u|^2 + (\phi\Lambda)^2}}{\Lambda}\right) + i\chi\right\},$$

and using this new coordinate, we can express the complex gauge coupling constant in the simple form

$$\frac{1}{g^2_{\rm YM}} + i\theta = \frac{\Delta\tilde{x}^6 + i\Delta x^{10}}{g_s\alpha'^{\frac{1}{2}}} = \frac{\Delta\tilde{\phi} + i\Delta\chi}{2}$$
$$= \ln y_+ - \ln y_-\,, \qquad (7)$$

where the $\ln y_\pm$ indicate the value of $\ln y$ at $x^6 = x^6_\pm$. This is the same form as that of pure SYM theory originally suggested in [18]. Therefore we can expect that the direct (supergravity) effect of the D6-branes on the four-dimensional field theory *disappears* by the coordinate transformation $x_6 \to \tilde{x}_6$ and the background *reduces to the same as pure SYM theory* in appearance. The effect of the D6-branes is implicitly included in $\Delta\tilde{x}_6$ and Δx_{10}.

In fact this new coordinate is one of the two holomorphic coordinates of (multi-)Taub-NUT space [27]. The above claim is consistent with the fact that the supersymmetric cycle written by this new coordinate reproduces the correct Seiberg–Witten curve for 4D $N=2$ SQCD.

The above observation is important when we consider the T-dualized system. When we take the T-duality in the direction of x_6, Δx_6 will change into the NSNS 2-form field $b^{\rm NS}$ coming from the twisted sector on the orbifold. In the case of pure SYM theory coupled with the general background, the gauge coupling constant[8] is known to be written as $e^{-\phi}b^{\rm NS}$ in combination with the dilaton e^{ϕ} [28].

In the previous attempts [16,17], the authors have used this formula and interpreted it as the holographic dual of the gauge coupling constant. This leads to a complicated and abnormal behavior of the twisted sector because of the non-trivial dilaton with logarithmic behavior.

But the above MQCD analysis casts some doubt on the applicability of this formula to this T-dualized (Type

[6] In the definition of u, we add the factor $\frac{1}{2\pi}$ to simplify the equations below

[7] There is another contribution coming from the RR 1-form $C^{\rm R}_6$. We can set the RR 1-form $C^{\rm R}_6$ as a constant, while keeping the non-trivial $C^{\rm R}_4$ and $C^{\rm R}_5$ which give the RR 2-form field strength in the D6 SUGRA solution (1). We set this constant $C^{\rm R}_6$ to zero below

[8] Here we use the term "gauge coupling constant" in a broad sense including the interaction with the background (dilaton). In [28], the quiver gauge theories are discussed and pure SYM theory is a special case of these

IIB) model. This analysis indicates that we have to take an appropriate "coordinate" instead of b^{NS}. Then the non-trivial $\mathrm{e}^{-\phi}$ is absorbed in this "coordinate" and gives no direct contribution to the behavior of the gauge coupling constant in appearance. In the following sections, we will discuss these matters by starting to study the behavior Δx_6 in the simpler case.

3 D2-branes and strings on D6 background

3.1 Single D2-brane with heavy quarks on N_f D6 background

Let us consider one D2-brane on the background of the N_f D6-branes. This is the T- and S-dual of a part of the MQCD configuration; one NS5-brane on the N_f D6 supergravity background. So it is useful to study the behavior of the D2-brane world-volume for our investigation of the 4D RG-flow.

In the same way as the previous section, the world-volume of the N_f D6-branes spans x^0, x^1, x^2, x^3, x^7, x^8, x^9 and they are located at $x^4=x^5=x^6=0$. We consider these D6-branes as the background described by the supergravity solution in the previous section. What happens if we put one D2-brane in this background? Let us consider the field theory on the D2-brane whose world-volume spans x^0, x^4 and x^5. This D2-brane is located at $x^7=x^8=x^9=0$, but delocalized in the directions of x^1, x^2 and x^3.

This type of the world-volume theory (the so-called Dp–D$(8-p)$ system) has been studied in various papers especially in the context of the string creation or the baryon vertex as wrapping D-branes [25,29–33]. In the (non-wrapping) D2–D6 system such as our model, it is known to be inappropriate for the analysis of string creation. This is because the asymptotic behavior is bad due to its fewer-dimensional world-volume. In [31], this D2–D6 system is discussed with the special care needed for the exceptional case.[9] Their analysis in the asymptotically flat background depends on the additional continuous parameter ν. In this discussion, this parameter has the origin in the partially wrapping D2-brane in the near-horizon limit of the background. On the other hand, the physical system must be realized only in the case with a special value for it; otherwise the RG-flow of the gauge coupling constant in our problem does not appear with the typical coefficient. So we need to check this point and show that it is true. As seen in the following, we can also show that in general there are N_f+1 inequivalent BPS configurations. This is the proof of the s-rule from the world-volume soliton. This gives the quantization of their parameter ν.

In addition to that, we want to know what will happen in the T-dualized version of the MQCD configuration. This D2–D6 system is the only configuration of the Dp–D$(8-p)$ systems which has a direct analogy with MQCD. So it is useful and necessary to treat this system purely in the Type IIA language, without mechanically using the analysis of the supersymmetric cycle in M-theory.

Let us study the D2-brane action on this background. The D2-brane can be described by the Born–Infeld action:

$$S_{\mathrm{D2}}= - T_{\mathrm{D2}}\int \mathrm{d}^3\sigma \mathrm{e}^{-\phi}\sqrt{-\det\left(g_{MN}\partial_\alpha X^M\partial_\beta X^N+2\pi\alpha' F_{\alpha\beta}\right)} +\frac{1}{2\pi g_{\mathrm{s}}\alpha'^{\frac{1}{2}}}\int G^R_{(2)}\wedge A_{(1)}, \tag{8}$$

where T_{D2} is the D2-brane tension $T_{\mathrm{D2}}=\left(4\pi^2 g_{\mathrm{s}}\alpha'^{\frac{3}{2}}\right)^{-1}$. We also denote the world-volume coordinates of the D2-brane $\sigma_\alpha=\{\sigma_0,\sigma_4,\sigma_5\}$ and M; N runs over all the ten dimension indices. Note that in the above equation we set the Chern–Simons term as the form in which the RR 2-form field strength appears instead of RR 1-form gauge field. This is because in the T-dual (T_6-dual) of this D2–D6 system, the anomaly cancellation requires that the Chern–Simons term on the D3-brane has the RR 1-form field strength, not the RR 0-form gauge field [34]. In the case of other ordinary backgrounds, the two kinds of form are the same up to a total derivative which has no physical meaning. But in the case such as this configuration, this total derivative gives the additional anomaly, which makes the total anomaly of this system zero. So the above form of the Chern–Simons term will be the correct form.

After taking the static gauge $\sigma_0=t$ and $\{\sigma_4,\sigma_5\}=\{x_4,x_5\}$, and assuming that only $X^6=X^6(x_4,x_5)$ and $A_0=A_0(x_4,x_5)$ are the non-trivial fields, let us take the limit (3). Then the above action reduces to[10]

$$\begin{aligned} S_{\mathrm{D2}}=&-\frac{\Lambda}{4\pi}\int \mathrm{d}t\mathrm{d}u^4\mathrm{d}u^5 \\ &\times H\left[\frac{1}{\Lambda^2}+\frac{1}{2}\left|\nabla_u\phi\right|^2-\frac{1}{2}\left|\nabla_u a\right|^2+O(\Lambda^2)\right] \\ &+\frac{\Lambda^2}{4\pi}\int \mathrm{d}t\mathrm{d}u^4\mathrm{d}u^5\frac{N_f}{R^2}\left(\frac{\phi}{R}-\nabla_u\phi\cdot\frac{\mathbf{u}}{R}\right)a. \end{aligned} \tag{9}$$

Here we use the convention in which the vector $\mathbf{u}$ indicates the two-dimensional vector in the space (u^4,u^5) and ∇_u indicates the derivative with respect to $\mathbf{u}$. We also denote by a the gauge field, $a=A_0/\Lambda$, and R as the rescaled length of r in the previous section such that $R\equiv\sqrt{|u|^2+(\Lambda\phi)^2}$. Note that this "length" contains the field ϕ which has a non-trivial dependence on (u_4,u_5). By using this R, H can be written as $H=1+(N_f\Lambda)/R$.

Let us add the source with $\pm N_c$ electric charge to the above action. In the context of string theory, this means that we add the semi-infinite N_c fundamental strings (F1) which are terminated on the D2-brane. The signs of $\pm N_c$ depend on the question on which side of the D3-brane they are terminated. We consider the source which is delocalized in the direction of x^1, x^2 and x^3 in order that our configuration has the isometry in these directions.

[9] An intensive study on the other Dp–D$(8-p)$ systems is also done in [31]

[10] Our analysis is limited within $O(\Lambda^2)$, but the result is the same if we start from the Born–Infeld action (see the appendix)

Naively, it seems that it is enough to add the source term to the previous action such that

$$\begin{aligned}\Delta S &= \pm N_c \int \mathrm{d}\sigma^3 A_0 \delta(x_4)\delta(x_5)\\ &= \pm N_c \Lambda \int \mathrm{d}t\mathrm{d}u_4\mathrm{d}u_5 \left\{ a\nabla_u \left(\frac{\mathbf{u}}{2\pi|u|^2} \right) \right\}.\end{aligned}$$

But the analysis of [35] suggests that we also need the source term for X^6 (or ϕ) in addition to the above source term for A_0 (or a). This additional source term makes the equation of motion consistent with the BPS condition[11]. By the results of these authors, we can determine the form and the coefficient of the additional source. This correct source term will be

$$\begin{aligned}\Delta S &= \pm N_c \int \mathrm{d}\sigma^3 \left\{ A_0 + (2\pi\alpha')^{-1} X^6 \right\} \delta(x_4)\delta(x_5)\\ &= \pm N_c \Lambda \int \mathrm{d}t\mathrm{d}u_4\mathrm{d}u_5 \left\{ (a+\phi)\, \nabla_u \left(\frac{\mathbf{u}}{2\pi|u|^2} \right) \right\}.\end{aligned}$$

From the action $S_{\mathrm{D2}} + \Delta S$, we can see the constraint for a (Gauss law),

$$\begin{aligned}\nabla_u \left(H \nabla_u a \right) &= \frac{N_f \Lambda}{R^3} \left(\phi - \nabla_u \phi \cdot \mathbf{u} \right) \pm 4\pi N_c \delta(u^4)\delta(u^5)\\ &= \nabla_u \left(-\frac{N_f \Lambda \phi}{R} \frac{\mathbf{u}}{|u|^2} + \{\mathrm{sign}(\phi_0) N_f \pm 2N_c\} \frac{\mathbf{u}}{|u|^2} \right),\end{aligned} \tag{10}$$

where ϕ_0 is the value of ϕ at $u = 0$. The term which includes the sign(ϕ_0) is needed to kill the delta function coming from the $\nabla_u \cdot \frac{\mathbf{u}}{|u|^2}$ of the first term.

When the D6-branes are located apart in the direction of x^6, the last term in the above equation is generalized to

$$\begin{aligned}&\nabla_u \Bigg(-\sum_{k=1}^{N_f} \frac{\Lambda(\phi - \phi_k)}{R_k} \frac{\mathbf{u}}{|u|^2}\\ &\qquad + \left\{ \sum_{k=1}^{N_f} \mathrm{sign}(\phi_0 - \phi_k) \pm 2N_c \right\} \frac{\mathbf{u}}{|u|^2} \Bigg),\end{aligned} \tag{11}$$

$$R_k^2 = |u|^2 + \Lambda^2 |\phi - \phi_k|^2 \quad (\phi_1 < \phi_2 < \ldots < \phi_{N_f}),$$

where $\Lambda\phi_k$ indicates the position of each D6-brane. So there are $N_f + 1$ choices for $\{\mathrm{sign}(\phi_0 - \phi_k)\}$, depending on the value of ϕ_0. As we will see, this leads to the $N_f + 1$ inequivalent BPS configurations.

The appearance of this sign term is the significant difference between the two choices about the form of the Chern–Simons term – which of the RR gauge field and the $U(1)$ gauge field we should keep as the gauge field, not the field strength. If we start by keeping the RR gauge field in the form of the gauge field, we will not have this sign(ϕ_0) term.

From the above equation, we can see the relation between a and ϕ,

$$H \nabla_u a = -\frac{N_f \Lambda \phi}{R} \frac{\mathbf{u}}{|u|^2} + \{\mathrm{sign}(\phi_0) N_f \pm 2N_c\} \frac{\mathbf{u}}{|u|^2}. \tag{12}$$

Using this relation, we can obtain the static Hamiltonian,

$$\begin{aligned}\mathcal{H} &= \frac{1}{4\pi\Lambda} \int (\mathrm{d}\mathbf{u})^2 H\\ &\quad \times \Bigg[1 + \frac{\Lambda^2}{2} |\nabla_u \phi|^2\\ &\qquad + \frac{\Lambda^2 H^{-2}}{2|u|^2} \left(\frac{N_f \Lambda \phi}{R} - N_f \mathrm{sign}(\phi_0) \mp 2N_c \right)^2 \Bigg]\\ &\quad \mp \int (\mathrm{d}\mathbf{u})^2 N_c \Lambda \phi \delta(\mathbf{u})^2\\ &\quad + \frac{\Lambda}{4\pi} \oint a \left(\frac{N_f \Lambda \phi}{R} - \mathrm{sign}(\phi_0) N_f \mp 2N_c \right) \frac{1}{|u|}.\end{aligned}$$

The last term is the boundary term on the circumference. When we set the radius of this circumference infinite, it gives a non-zero contribution with the factor $\Lambda(\mp N_c - \mathrm{sign}(\phi_0) N_f/2)$ to the static Hamiltonian. In the case of $N_c = 0$, this is the same as the contribution coming from the R-sector in the open string one-loop amplitude.

Next, let us consider the equation of motion. We can easily obtain

$$\begin{aligned}&\nabla_u \left(H \nabla_u \phi \right)\\ &= \frac{N_f \Lambda H^{-1}}{R^3} \left\{ -\phi - \mathrm{sign}(\phi_0) N_f \mp 2N_c \right\}\\ &\quad + \frac{1}{2} \frac{\partial H}{\partial \phi} \Bigg[(\nabla_u \phi)^2\\ &\qquad - \frac{H^{-2}}{|u|^2} \left(-\frac{N_f \Lambda \phi}{R} + N_f \mathrm{sign}(\phi_0) \pm 2N_c \right)^2 \Bigg]\\ &\quad \mp 4\pi N_c \delta^2(\mathbf{u})\\ &= -\frac{N_f \Lambda}{R^3} \left\{ \phi + (\nabla_u a \cdot \mathbf{u}) \right\}\\ &\quad + \frac{1}{2} \frac{\partial H}{\partial \phi} \left[(\nabla_u \phi)^2 - (\nabla_u a)^2 \right] \mp 4\pi N_c \delta^2(\mathbf{u}).\end{aligned} \tag{13}$$

To get the last equation, we have used the relation (12). From this form, we can see that if there is the additional relation $\nabla a = -\nabla \phi$, the above equation will be satisfied by the Gauss law (10). So we can expect that this is the "almost" BPS condition.[12] As a result of that, we obtain

[11] That analysis is limited in the region which is far from the source, so this source term does not appear explicitly in the discussion

[12] The same form of the "almost" BPS condition has appeared in [29–32, 35] in which the world-volume soliton or the string creation is discussed

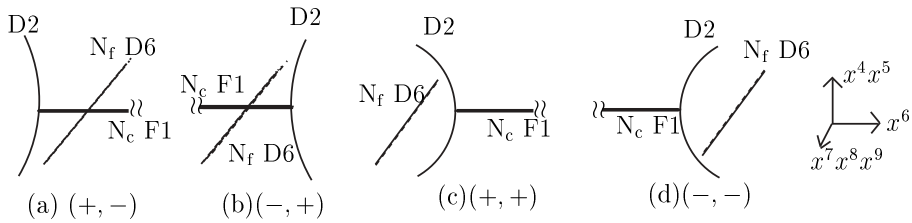

Fig. 1a–d. A rough sketch for the configuration according to the combinations of (sign$\{\pm 2N_c\}$, sign$\{\phi_0\}$)

the equation to determine the behavior of ϕ,

$$H\nabla_u\phi = \frac{N_f\Lambda\phi}{R}\frac{\mathbf{u}}{|u|^2} - \{\mathrm{sign}(\phi_0)N_f \pm 2N_c\}\frac{\mathbf{u}}{|u|^2}. \quad (14)$$

Using the good "coordinate" $\tilde{\phi}$ which corresponds to the new coordinate $\tilde{x}^6/(2\pi\alpha'\Lambda)$ in Sect. 2, the above equation gives us the solution

$$\tilde{\phi} \equiv \phi + N_f \ln\left(\frac{R+\Lambda\phi}{\Lambda}\right) \quad (15)$$
$$= -\{\pm 2N_c - N_f + \mathrm{sign}(\phi_0)N_f\}\ln\left(\frac{|u|}{\Lambda}\right) + \mathrm{const.}$$

We can generalize the above result to the case in which the D6-branes are located apart in the direction of x^6:

$$\tilde{\phi} \equiv \phi + \sum_{k=1}^{N_f}\ln\left(\frac{R_k + \Lambda(\phi-\phi_k)}{\Lambda}\right)$$
$$= -\left\{\pm 2N_c - N_f + \sum_{k=1}^{N_f}\mathrm{sign}(\phi_0-\phi_k)\right\}\ln\left(\frac{|u|}{\Lambda}\right)$$
$$+\mathrm{const.} \quad (16)$$

This shows that there are N_f+1 inequivalent BPS configurations depending on the value of ϕ_0. This corresponds to the s-rule [24].[13] These BPS configurations cannot be continuously transformed into each other.

In addition to the bulk equation of motion, we have to consider the variation of the boundary term. We obtain at infinity,

$$\delta\phi\,(\nabla_u\phi\cdot\mathbf{u}) - \delta a\,(\pm 2N_c + \mathrm{sign}(\phi_0)N_f) = 0. \quad (17)$$

This shows us that ϕ never becomes a constant even at infinity. In fact we obtain the logarithmic flow of ϕ from (15):

$$\phi \sim -(\pm 2N_c + \mathrm{sign}(\phi_0)N_f)\ln|u|. \quad (18)$$

This behavior is consistent with the above boundary condition.

[13] In [25], they have confirmed that there is the maximum of N for the continuous parameter ν corresponding to the string charge in the D3–D5 system. This is one aspect of the s-rule, while the original s-rule [24] is the statement that there are $N+1$ inequivalent BPS configurations in this model. Our result shows the quantization of this parameter ν from the point of view of these authors

In the following sections, we consider the case of $2N_c \geq N_f$ which corresponds to the asymptotic free or conformal gauge theory as we will see.[14] In this case, we can make the rough sketch Fig. 1 for the configuration according to the sign of $\pm 2N_c$ and sign(ϕ_0) in (15).

3.2 Fundamental string charge and eleventh dimension

It is well known that the electric charge on the D-brane corresponds to the fundamental string charge. In this section, we will discuss the relation between the two kinds of charge in our model to connect the gauge field to the eleventh dimension. First, we can rewrite the Gauss law (10) as follows:

$$\nabla_u\left\{H\nabla_u a + N_f\left(\frac{\Lambda\phi}{R} - \mathrm{sign}(\phi_0)\right)\frac{\mathbf{u}}{|u|^2}\right\}$$
$$= \pm 4\pi N_c\delta(u^4)\delta(u^5). \quad (19)$$

Remember that from the action (9), the electric charge $\mathrm{Q_E}$ is given by the integral of the left hand side:

$$\mathrm{Q_E} = \frac{1}{4\pi}\oint *\left\{H\nabla_u a + N_f\left(\frac{\Lambda\phi}{R} - \mathrm{sign}(\phi_0)\right)\frac{\mathbf{u}}{|u|^2}\right\}$$
$$= \pm N_c, \quad (20)$$

where $*$ means the dual in the 2D space, and this integral is calculated over the circle at the fixed $|u|$. The right hand side of this equation indicates that only the explicit external $\pm N_c$ F1 source is the total electric charge. This is consistent with the fact that, in our model, there is only a $\pm N_c$ F1 source from the starting point. But what is the meaning of the left hand side? The second term is the Witten effect coming from the Chern–Simons term. The above result shows that this term gives an additional induced charge, but it is canceled by the non-trivial contribution from $H\nabla_u a$. In order to give this additional contribution, the gauge field a turns out to show a non-trivial behavior. This makes ϕ non-trivial because a and ϕ are related by supersymmetry ("almost" BPS condition). This is the dielectric effect similar to Myer's effect [36] which occurs in another supersymmetric configuration like the D6(123789)–D2(89) system.[15] The definition of the electric current is different according to the sign of ϕ_0. This

[14] In the case of $N_f > 2N_c$, most of our method can be applicable except in the ultra-violet region

[15] By using a D-brane wrapped on a sphere, the interpretation as Myer's effect is also given in [32]

difference produces the interpretation of the string creation or the Hanany–Witten effect. Note that the observer on the D2-brane never sees such a string creation because there is only an external $\pm N_c$ electric charge on the D2-brane. But this relative difference is important for the whole system and we need a definition of the current which is applicable for the whole system.

So let us take the current of $\phi_0 > 0$ as the standard. Then we define the "dual" field as

$$\tilde{\nabla}_u \chi \equiv H \nabla_u a + N_f \left(\frac{\Lambda \phi}{R} - 1 \right) \frac{\mathbf{u}}{|u|^2} \quad ,$$
$$\tilde{\nabla}_u \equiv \begin{pmatrix} -\partial_5^u \\ \partial_4^u \end{pmatrix}. \tag{21}$$

By this definition, we can change the dynamical variable from (a, ϕ) to (χ, ϕ). This χ can measure the difference of the electric charge, namely the fundamental string charge. This means that χ can also measure the distance of the x^{10}-direction and that we can identify this χ with that appearing in Sect. 2.

The classical solution for χ can be obtained from its definition and the Gauss law:

$$\chi = [\mp 2N_c + N_f \{1 - \mathrm{sign}(\phi_0)\}] \varphi + \mathrm{const.} \quad , \tag{22}$$

where we define φ by $u = u^4 + \mathrm{i}u^5 = |u|\mathrm{e}^{\mathrm{i}\varphi}$. By combining this solution with that of ϕ, we obtain the solution expressed in the complex "coordinate" y appearing in Sect. 2 as

$$\ln y = \frac{\tilde{\phi} + \mathrm{i}\chi}{2} = \frac{1}{2} \left\{ \phi + \mathrm{i}\chi + N_f \ln \left(\frac{R + \Lambda\phi}{\Lambda} \right) \right\} \tag{23}$$
$$= \mp N_c \ln \left(\frac{u}{\Lambda} \right) + \frac{N_f}{2} \{1 - \mathrm{sign}(\phi_0)\} \ln \left(\frac{u}{\Lambda} \right) + \mathrm{const.}$$

In addition to y, let us define another complex variable w,

$$w \equiv \mathrm{e}^{-(\phi + \mathrm{i}\chi)/2} \left(\frac{u}{|u|} \right)^{N_f} \left(\frac{-\phi\Lambda + R}{\Lambda} \right)^{N_f/2}. \tag{24}$$

Then we can express the above solution by using these complex variables as follows:

$$y = \left(\frac{u}{\Lambda} \right)^{\mp N_c} \times \mathrm{const.} \qquad \phi_0 > 0,$$
$$w = \left(\frac{u}{\Lambda} \right)^{\pm N_c} \times \mathrm{const.} \qquad \phi_0 < 0.$$

This expresses the holomorphic embedding in the four-dimensional space with an $A_{N_f - 1}$ singularity, $yw = (u/\Lambda)^{N_f}$. This is the result expected from the analysis when we lift our model to M-theory and study the supersymmetric cycle of the M2-brane on the multi-Taub-NUT background with coincident N_f monopoles.

Note that, until now, we have considered the case in which the external N_c strings have infinite lengths. This means that the source with the N_c charges is too heavy to have the dynamics. This is why our analysis in Type IIA theory agrees with that of M-theory. If these strings are not infinite, that is, the source is not so heavy, there will be some dynamical effect as happens in the case of MQCD. Therefore the analysis in Type IIA theory is limited within the approximation in which we can ignore this effect.

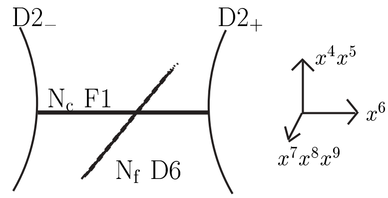

Fig. 2. The combination of **a** and **b** of Fig. 1

3.3 Two D2-branes with heavy quarks on D6 background

Let us generalize the previous result to the case with two D2-branes and N_c fundamental strings stretching between them. We consider the configuration in which the N_f background D6-branes are located between the two D2-branes. This is a situation similar to the MQCD configuration. We can see that $U(1)_- \times U(1)_+$ gauge theory with heavy N_c bifundamental quarks on a N_f D6 background is realized on this configuration.[16] Here, we distinguish each D2-brane and $U(1)$ factor by $\pm$. This corresponds to the situation that the $U(2)$ gauge theory is broken into the $U(1)_- \times U(1)_+$ gauge theory by the relative difference between the non-trivial fields $\tilde{\phi}_\pm$ on the D2-branes. Then, we treat the two D2-branes almost independently, except that in this situation, the signs of $\pm 2N_c$ and $\mathrm{sign}(\phi_0)N_f$ appear in the combination $(-2N_c, N_f)$ and $(+2N_c, -N_f)$. (see Figs. 1 and 2.)

This leads to the insight that the relative distance $\Delta\{\ln y\} \equiv (\ln y_+ - \ln y_-)$ is determined by

$$\Delta\{\ln y\} = \ln y_+ - \ln y_- = \frac{1}{2} \left\{ \Delta\tilde{\phi} + \mathrm{i}\Delta\chi \right\}$$
$$= (2N_c - N_f) \ln \frac{u}{\Lambda} + \mathrm{const.} \tag{25}$$

This is the correct behavior for the RG-flow of $SU(N_c)$ SQCD with N_f flavors.

4 T-dualized configuration

Until now, we have discussed the configuration which is analogous to the MQCD configuration. We have studied the 2+1 dimensional field theory realized on this configuration. In this section, we discuss the T-dualized configuration in

[16] This is a rough approximation and we know that there is a non-perturbative effect. But here, we proceed keeping in mind that this approximation is justified only in the ultra-violet region $u \gg \Lambda_{\mathrm{QCD}}$ (QCD scale) or in the large N_c limit. We will come back to this problem later

the direction of x^6. For our original purpose, we have to realize the equivalent $2+1$ dimensional field theory also on the T-dualized configuration. How does the previous D2–D6 system transform under T-duality? Naively it seems to be D3-branes on the background of the D7 SUGRA solution. But there might be some confusion about what is the background after the T-duality.

In the previous attempts [16, 17], these authors have considered the Type IIB configuration with the orbifold on the D7 SUGRA background. They have regarded this configuration as the T-dual of the MQCD configuration with two NS5-branes on the D6 SUGRA background. As a result, the logarithmic behavior of the D7 SUGRA solution makes the analysis very messy. This also hampers the AdS_5 structure in the conformal case, as found in [16]. This logarithmic behavior is the origin of the abnormal (complicated) behavior of their result. This strongly suggests that the D7 SUGRA background will not be the correct background as the T-dual of the MQCD configuration.

The most important point is that the region where the D7 classical solution is effective corresponds to that of the MQCD configuration with small x^6-radius. In this region, the two NS5-branes are wrapping this direction and crossing each other. We cannot expect that the ordinary 4D gauge theory is realized on this configuration. So it is unlikely that the role of the D6 SUGRA solution as the background will simply be succeeded by the D7 SUGRA solution.

In other words, the background in Type IIB theory must have the radius R^{IIB} which satisfies the relation $R_6^{\mathrm{IIA}} = \alpha'/R_6^{\mathrm{IIB}} \to \infty$. This is the situation for the D6 background in the MQCD configuration. In this sense, the D7 SUGRA solution is not equivalent to the D6 SUGRA background in the MQCD configuration. The D7-brane solution is obtained from the small R^{IIA} limit of the D6 solution. On the other hand, this requirement is satisfied in the case of pure SYM theory; the backgrounds are flat before and after the T-duality.

Also in our simplified model (T-dual of the D2–D6 system), the D7 SUGRA background will not be the correct background. But it is very plausible that the scalar field $X^6/2\pi\alpha'$ on the D2-brane world volume is transformed into the Wilson line (or gauge field) A_6 on the D3-brane. Then, the form of the D3-brane action after such a translation enables us to guess at least what the background is which interacts with the fields on the D3-brane.

Let us look back at the D2-brane action (9) and consider how the action will change after the plausible T-duality. After rewriting $(\phi,\, a)$ by $(a_6,\, a_0)$, where $a_6 \equiv \frac{A_6}{\Lambda}$ and $a_0 \equiv \frac{A_0}{\Lambda}$, we expect that the D2-brane action reduces to that of the D3-brane with the delocalized direction of x^6. The D3-brane action after the dimensional reduction in this direction will be

$$
\begin{aligned}
S_{\mathrm{D3}} = &-\frac{\Lambda}{4\pi}\int \mathrm{d}t\mathrm{d}u^4\mathrm{d}u^5 \\
&\times H\left[\frac{1}{\Lambda^2} + \frac{1}{2}\left|\nabla_u a_6\right|^2 - \frac{1}{2}\left|\nabla_u a_0\right|^2 + O(\Lambda^2)\right] \\
&+ \frac{\Lambda^2}{4\pi}\int \mathrm{d}t\mathrm{d}u^4\mathrm{d}u^5 \frac{N_f}{R^2}\left(\frac{a_6}{R} - \nabla_u a_6 \cdot \frac{\mathbf{u}}{R}\right) a_0, \qquad (26)
\end{aligned}
$$

where we define R and H as $R \equiv \sqrt{|u|^2 + (\Lambda a_6)^2}$ and $H \equiv 1 + \frac{\Lambda N_f}{R}$.

Let us estimate the form of the "background" on the D3-brane which gives the D3-brane action (26). It is known that the D3-brane action on the general background can be written as

$$
\begin{aligned}
S_{\mathrm{D3}} = \\
&- T_{\mathrm{D3}}\int \mathrm{d}^4\sigma \mathrm{e}^{-\phi}\sqrt{-\det\left(g_{MN}\partial_\alpha X^M \partial_\beta X^N + 2\pi\alpha' F_{\alpha\beta}\right)} \\
&+ \frac{1}{4\pi^2 g_{\mathrm{s}}\alpha'}\int G^{\mathrm{R}}_{(3)} \wedge A_{(1)}, \\
G^{\mathrm{R}}_{(3)} \equiv\ & \mathrm{d}B^{\mathrm{R}}_{(2)} + \left(B^{\mathrm{NS}}_{(2)} + 2\pi\alpha' F_{(2)}\right) \wedge \mathrm{d}C^{\mathrm{R}}_{(0)}, \qquad (27)
\end{aligned}
$$

where we defined T_{D3} as the D3-brane tension $T_{\mathrm{D3}} = (8\pi^3 g_{\mathrm{s}}\alpha'^2)^{-1}$. In the above equation, $B^{\mathrm{R}}_{(2)}, B^{\mathrm{NS}}_{(2)}$ and $C^{\mathrm{R}}_{(0)}$ are the RR 2-form, NSNS 2-form and RR 0-form gauge field respectively. First, we have to be careful with the region where the field theory is a good description. Let us denote the radius in the direction of x^6 in Type IIB theory as R_6 $(= R_6^{\mathrm{IIB}})$ for simplicity. By using the relation of the string coupling constant[17] between before (IIA) and after (IIB) the T-duality, $g_s^A = g_s^B \alpha'^{1/2}/R_6$, we can express Λ as $\Lambda = g_s^A/\left(4\pi\alpha'^{1/2}\right) = g_s^B/(4\pi R_6)$. Then in Type IIB theory, we can take the limit similar to (3):

$$
\begin{gathered}
g_{\mathrm{s}} \to 0\,, \quad R_6 \to 0\,, \quad z \to 0\,, \quad \alpha' \to 0\,; \\
\Lambda \equiv \frac{g_{\mathrm{s}}}{4\pi R_6}\,, u \equiv \frac{z}{2\pi\alpha'} \quad \text{fixed}\,. \qquad (28)
\end{gathered}
$$

In addition to the above limit, let us take the limit $\alpha'/R_6 \to \infty$. This means that the compactified radius of the x^6-direction becomes infinite in Type IIA theory. This is the same situation as in the previous sections.

Then we can easily read off the "background" from the actions (26) and (27) in the limit of (28). The "background" is written

$$
\begin{aligned}
\mathrm{d}s^2_{10} = &H^{-\frac{1}{2}}\left\{\eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + \sum_{i=7,8,9}(\mathrm{d}x^i)^2 + (\mathrm{d}x^6)^2\right\} \\
&+ H^{\frac{1}{2}}|\mathrm{d}z|^2, \\
\mathrm{e}^{-\phi} = g_{\mathrm{s}}^{-1}H, &\qquad \partial_4 B^{\mathrm{R}}_{56} - \partial_5 B^{\mathrm{R}}_{46} = \frac{N_f a_6 \Lambda^2}{2\pi\alpha' R^3}, \qquad (29) \\
\nabla C^{\mathrm{R}}_{(0)} = &\frac{N_f\Lambda}{2\pi\alpha' R^3}\begin{pmatrix}-u^5 \\ u^4\end{pmatrix}, \qquad B^{\mathrm{NS}}_{64} = B^{\mathrm{NS}}_{56} = 0,
\end{aligned}
$$

[17] We have used the same symbol g_{s} to express the string coupling constant for both IIA and IIB theory. Here we distinguish the two kinds of string coupling constants

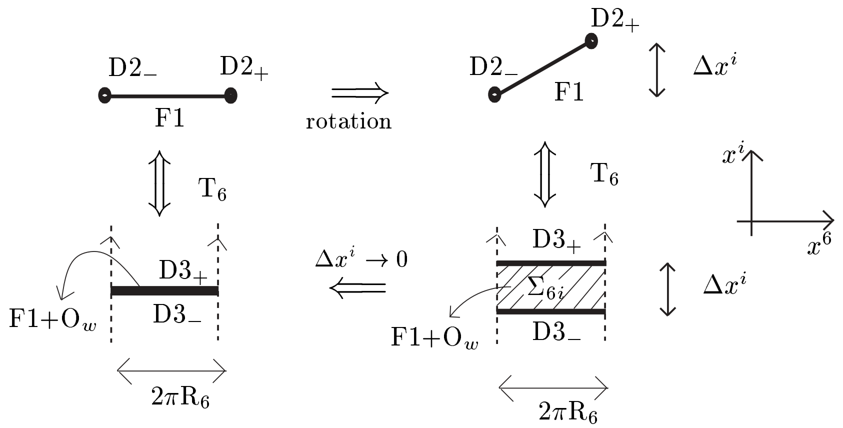

Fig. 3. The sketch in the case of $N_f = 0$. The number of Type IIA fundamental strings transforms to that of Type IIB fundamental strings. They are stretching over the vanishing distance expressed as the zero limit of the small resolution Δx^i

where we take the static gauge for the action (27) as before. Here we define ∇ as the derivative with respect to the original coordinates of the 2D space (x^4, x^5). Note that in the above expression, the above "background" expresses the only gravitational field on the D3-brane. We cannot guess how the background behaves in the bulk away from the D3-brane. But we have to remember that only the geometry near the brane is important for the AdS/CFT correspondence. So it is enough for that purpose to obtain the information about the background on the brane.

Note that the inclusion of the Wilson line with periodicity means that this expression contains all the winding modes of the x^6-direction. This is equivalent to the fact that the D6 supergravity background contains all the Kaluza–Klein (KK) modes in this direction. As the radius R_6 becomes large, the effect of the non-zero winding modes drops and we have to change the warped factor:

$$\frac{\Lambda N_f}{R} \to \sum_{n=-\infty}^{\infty} \frac{\Lambda N_f}{\sqrt{|u|^2 + (\Lambda a_6 + n/R_6)^2}} \overset{|u| \gg 1/R_6}{\sim} \frac{g_s N_f}{2\pi} \ln |u| + \text{const.}. \tag{30}$$

Then the above "background" becomes the simple D7 supergravity solution with the non-trivial dilaton and RR 0-form. It is this simple D7 solution that has been used in [16, 17] as the background. But in the region with the large radius R_6 (small radius α'/R_6 in Type IIA), we cannot expect any more that the result of the MQCD analysis will be reproduced in Type IIB theory. This will be the reason why their result seems to be different from the result expected from the 4D field theory.

Here, we have to comment on the anomaly in-flow mechanism [34] in the D3–D7 system with the two-dimensional intersection. This means the cancellation between the anomaly coming from the chiral fermion on the intersection and the anomaly from the bulk Chern–Simons term. In our model, the fermion one-loop effect is included in the first term of (27) and the bulk Chern–Simons term corresponds to the second term. As for the first term, this is the picture of the closed string. These two contributions are canceled under the BPS condition such as $\nabla a_0 = -\nabla a_6$ and the Gauss law.

Next, let us discuss the behavior of the field on this D3-brane. We can repeat the same procedure as before by replacing (ϕ, a) and ϕ_0 in Sect. 3 with (a_6, a_0) and $a_6|_{u=0}$. In the same way, we define the field which expresses the F1 density on the D3-brane by

$$\tilde{\nabla}_u \chi \equiv H \nabla_u a_0 + N_f \left(\frac{\Lambda a_6}{R} - 1 \right) \frac{\mathbf{u}}{|u|^2},$$
$$\tilde{\nabla}_u \equiv \begin{pmatrix} -\partial_5^u \\ \partial_4^u \end{pmatrix}. \tag{31}$$

What is the fundamental string like whose charge is described by the above field χ ? Let us consider the case with two D3-branes and $\pm N_c$ additional electric (F1) source on the flat background ($N_f = 0$). This also expresses the $U(1)_- \times U(1)_+$ gauge theory with N_c bifundamental matter particles as discussed in the previous sections. The source is realized as the T-dual of the fundamental string in the previous section. This fundamental string in Type IIB theory is stretching over the vanishing distance between the two D3-branes, say, the distance in the direction x^i ($i = 7, 8, 9$). As an example, the rough sketch is depicted in Fig. 3. On the other hand, the information about the distance between the D2-brane is transferred to the integral on the vanishing two-cycle $\Sigma(6i)$ of the NSNS 2-form field $\int_{\Sigma(6i)} B^{\mathrm{NS}}_{(6i)}$. This integral is also the holographic charge[18] of the wave O_w per fundamental string mentioned in the above.

In the case with $N_f \neq 0$, there are also N_c fundamental strings stretching over the vanishing distance. But different from the case with $N_f = 0$, we have to generalize the NSNS 2-form field in the same way as $\Delta\tilde{\phi}$ in the previous section. This generalized NSNS 2-form field gives the correct wave O_w charge per fundamental string. These topics

[18] Here "holographic charge" means the charge which is observed at u in the same way as the RG-flow of the gauge coupling constant in Sect. 2. This charge shows how the string winds around in the direction of x^6

will be discussed in the following sections. Here we limit our analysis in this section within that of the world-volume theory and proceed.

Assuming that the bifundamental matter particles are too heavy to give a dynamical effect, we can handle the two D3-branes almost independently. Then by using the complex coordinate y similar to the previous sections,

$$\ln y \equiv \frac{\tilde{a}_6 + \mathrm{i}\chi}{2} = \frac{1}{2}\left\{a_6 + \mathrm{i}\chi + N_f \ln\left(\frac{R + \Lambda a_6}{\Lambda}\right)\right\}, \tag{32}$$

we obtain the solutions and the difference between them as

$$\ln y_\pm = \pm\left\{N_c - \frac{N_f}{2}\right\}\ln\left(\frac{u}{\Lambda}\right) + \frac{N_f}{2}\ln\left(\frac{u}{\Lambda}\right) + \text{const.}$$

$$\begin{aligned}\Delta\{\ln y\} &= \ln y_+ - \ln y_- = \frac{1}{2}\left(\Delta\tilde{a}_6 + \mathrm{i}\Delta\chi\right)\\ &= (2N_c - N_f)\ln\left(\frac{u}{\Lambda}\right) + \text{const.}\end{aligned} \tag{33}$$

5 From the fields on the brane to the fields of supergravity

In the previous section, we have learned that the (generalized) rescaled Wilson line a_6 ($\tilde{a}_6$) and the new field χ on the D3-brane give the non-trivial solution due to the background in the action (26). Let us consider rewriting these non-trivial fields in terms of the Type IIB SUGRA matter (gauge) fields. This will be useful for the application to the AdS/CFT (gravity/field theory) correspondence. This is also the necessary procedure, because this teaches us how to transform these field under the sequence of T- and S-dualities.

What are the supergravity matter (gauge) fields corresponding to a_6 and χ? First, let us consider the rescaled Wilson line a_6. It is well known that the Wilson line on one D-brane can be measured by the string world sheet coupled by a NSNS 2-form field. This world sheet spans the circle of the compactified direction (x^6-direction in our case) and the orthogonal semi-infinite line from the point located by the D-brane. From the field theoretical point of view, a string stretching on this semi-infinite line expresses an external heavy quark on the D-brane. Then we can obtain the Wilson line by the integral of the gauge field over the (compactified) circle, that is, by the world sheet with NSNS 2-form field.

Let us discuss how to express the rescaled Wilson line a_6 in our model. First, let us consider x^i as one of the coordinates (x^7, x^8, x^9). They are in the orthogonal directions to the D3-brane in the previous section. We have the relation between the NSNS field and the Wilson line:

$$\frac{1}{2\pi\alpha'}\int_0^{2\pi \mathrm{R}_6}\int_{x^i=\infty}^{x^i=0}\mathrm{d}x^6\mathrm{d}x^i B_{6i} = \oint_{x^i=0}\mathrm{d}x^6 A_6 - \oint_{x^i=\infty}\mathrm{d}x^6 A_6, \tag{34}$$

where we denote by $x^i=0$ the position of the D3-brane in this direction. We set the Wilson line at infinity zero below. Remember that we consider the location of the N_f D6-branes as the origin in the Type IIA analysis. This means a vanishing Wilson line for the background.

Then we have a relation between a_6 on the D3-brane and the NSNS field:

$$\frac{g_\mathrm{s} a_6}{2} = \oint_{x^i=0}\mathrm{d}x^6 A_6 = \frac{1}{2\pi\alpha'}\int_0^{2\pi \mathrm{R}_6}\int_{x^i=\infty}^{x^i=0}\mathrm{d}x^6\mathrm{d}x^i B_{6i}^{\mathrm{NS}}. \tag{35}$$

Let us define new fields, b^{NS} and $b^{\tilde{\mathrm{NS}}}$ as the NS fields corresponding to $(g_\mathrm{s}\Delta a_6)/2$ and $(g_\mathrm{s}\Delta\tilde{a}_6)/2$ in the previous section. They can be written as

$$\begin{aligned} b^{\mathrm{NS}} &\equiv \frac{1}{2\pi\alpha'}\int_{\Sigma(6i)} B_{6i}^{\mathrm{NS}}\mathrm{d}x^6\mathrm{d}x^i = \oint A_6^+\mathrm{d}x^6 - \oint A_6^-\mathrm{d}x^6\\ &= \frac{g_\mathrm{s}}{2}\Delta a_6,\\ \tilde{b}^{\mathrm{NS}} &\equiv \frac{1}{2\pi\alpha'}\int_{\Sigma(6i)} \tilde{B}_{6i}^{\mathrm{NS}}\mathrm{d}x^6\mathrm{d}x^i \equiv \frac{g_\mathrm{s}}{2}\Delta\tilde{a}_6\\ &= b^{\mathrm{NS}} + N_f\ln\left(\frac{\mathrm{R}_+ + \Lambda a_{6+}}{\mathrm{R}_- + \Lambda a_{6-}}\right),\end{aligned}$$

where $\frac{1}{2\pi\alpha'}$ is the normalization factor for the two-cycle $\Sigma(6i)$. As in the previous section, this two-cycle spans the circle with the radius R_6 and the vanishing distance between the two D3-branes in the x^i-direction (see Fig. 3). In the above, we distinguish the gauge field A_6, a_6 and R on each D3-brane by giving $+$ or $-$ on it. Note that as seen from these equations, B_{6i}^{NS} and $\tilde{B}_{6i}^{\mathrm{NS}}$ have only a x^i dependence as the delta function and a non-trivial (x^4, x^5) dependence.

Next, let us discuss what the supergravity fields are corresponding to $\Delta\chi$. Remember that χ measures the difference of the string density on the D3-brane. As discussed in Sect. 4, there are N_c fundamental strings stretching over the vanishing distance between the two D3-branes in the direction of x^i ($i=7,8,9$). Then we can see that the NSNS 2-form related to the F1-charge will be the one corresponding to $\Delta\chi$. From the equation which gives the F1-charge on (x^4, x^5) space, we get

$$\begin{aligned}&\frac{1}{4\pi}\partial_\alpha\left(\Delta\chi\right)\\ &= \frac{1}{(4\pi^2\alpha')^3}\int\left(*H_{(3)}^{\mathrm{NS}}\mathrm{e}^{-\phi}\right)_{\alpha 1236jk}\mathrm{d}x^{123}\mathrm{d}x^6\mathrm{d}x^{jk}\\ &= \frac{1}{(4\pi^2\alpha')^3}\int H_{\alpha 1236jk}^{\mathrm{NS}}\mathrm{d}x^{123}\mathrm{d}x^6\mathrm{d}x^{jk},\end{aligned}$$

where the indices $\{j,k\}$ are in $\{7,8,9\}$, but $\{j,k\}\neq i$, and the index α is in $\{4,5\}$. In the above, $\frac{1}{(4\pi^2\alpha')^3}$ is the normalization to give the integer F1-charge. We also use the last expression as the Poincaré dual of the NS-NS 3-form field strength. In the T-dualized (Type IIB) model, the directions of $\{x^{0123}, x^6\}$ are delocalized.[19] So we can see that this 7-form $H_{\alpha 1236jk}^{\mathrm{NS}}$ has a (x^j, x^k) dependence as

[19] As we will see, we will take the T-duality in these directions

the delta function in addition to the non-trivial (x^4, x^5) dependence.

Then we can write down the solution in terms of the fields of supergravity as follows:

$$\begin{aligned}\frac{1}{2}\{\Delta\tilde{a}_6+\mathrm{i}\Delta\chi\} &= \frac{1}{2\pi\alpha'}\int_{\Sigma(6i)} g_{\mathrm{s}}^{-1}\tilde{B}^{\mathrm{NS}}_{6i}\mathrm{d}x^6\mathrm{d}x^i \\ &\quad +\frac{\mathrm{i}}{(2\pi)^5\alpha'^3}\int B^{\mathrm{NS}}_{1236jk}\mathrm{d}x^{123}\mathrm{d}x^6\mathrm{d}x^{jk} \\ &= (2N_c-N_f)\ln u+\text{const.} \quad , \end{aligned} \tag{36}$$

where the NSNS 6-form gauge field B^{NS}_{1236jk} is defined as $\partial_\alpha B^{\mathrm{NS}}_{1236jk}=H^{\mathrm{NS}}_{\alpha 1236jk}$. Note that the above supergravity gauge fields are living only "between" the two overlapping D3-branes and are similar to those of the twisted sector on the orbifold. But we have to be careful because of the fact that the above real part is not written only by $\frac{1}{2\pi\alpha'}\int_{\Sigma(6i)}\mathrm{e}^{-\phi}B^{\mathrm{NS}}_{6i}$, which is different from the case of pure SYM theory. Note that the integral

$$\frac{1}{2\pi\alpha'}\int_{\Sigma(6i)} g_{\mathrm{s}}^{-1}\tilde{B}^{\mathrm{NS}}_{6i}\mathrm{d}x^6\mathrm{d}x^i$$

is also the holographic wave charge per fundamental string.

It will be interesting to bring our results to the configuration in which the four-dimensional gauge theory is realized. This is the topic of the next section.

6 4D $\mathcal{N}=2$ field theory and gravity solution

6.1 Gauge coupling constant and 2-form fields

Let us take T-dualities and S-dualities of our configuration, and make the model in which 4D $\mathcal{N}=2$ SQCD is realized. This is the T-dualized model obtained from the well-known MQCD configuration mentioned in Sect. 2 and our result will give us some knowledge of what it is like.

We consider the sequences of the dualities, $\mathrm{T}_{36}\mathrm{S}\mathrm{T}_{12}\mathrm{S}\mathrm{T}_{36}$, where the indices mean the directions in which we take the T-dualities. Remember that the directions $\{x_1,x_2,x_3\}$ do not play any active role in our analysis. On the D2 worldvolume, the three scalar fields for these directions are free and decoupled from the remaining interacting action (9). In fact, we can easily confirm that there is no warped factor H in their kinetic terms. So we can safely delocalize these directions without changing our analysis. This is also the reason why the two-dimensional supersymmetric cycle for the M2-brane on the Taub-NUT background is the same as that of the M5-brane on the same background. As for the direction of x_6, we have already discussed the T-duality in this direction with special care in Sect. 4. The other directions $\{x_4, x_5, x_7, x_8, x_9\}$ are important for the structure of the vacua of the 4D field theory, but we do not take the T-dualities of these directions.

Let us consider what the constituents in our model will transform into. They are expected to transform under these dualities as

$$\begin{aligned} &\frac{1}{2\pi\alpha'}\int_{\Sigma(6i)} g_{\mathrm{s}}^{-1}\tilde{B}^{\mathrm{NS}}_{6i}\mathrm{d}x^6\mathrm{d}x^i \\ &\quad\to \frac{1}{2\pi\alpha'}\int_{\Sigma(6i)} g_{\mathrm{s}}^{-1}\tilde{B}^{\mathrm{NS}}_{6i}\mathrm{d}x^6\mathrm{d}x^i, \\ &\frac{1}{(2\pi)^5\alpha'^3}\int_{\Sigma(jk)} B^{\mathrm{NS}}_{1236jk}\mathrm{d}x^{123}\mathrm{d}x^6\mathrm{d}x^{jk} \\ &\quad\to \frac{1}{2\pi\alpha'}\int_{\Sigma(jk)} B^{\mathrm{R}}_{jk}\mathrm{d}x^{jk}, \\ &\mathrm{D3}(456)\to \text{Kaluza–Klein monopole}(12345), \\ &N_c\ \ F1(i)\to N_c\ \ D5(1236i), \\ &O_w(6)\to \mathrm{D3}(123). \end{aligned} \tag{37}$$

In the above, we can see how the new NSNS 2-form $\tilde{B}^{\mathrm{NS}}_{6i}$ transforms by (35) and the property $B^{\mathrm{NS}}_{6i}\to B^{\mathrm{NS}}_{6i}$ under this transformation. We also have to mention that the background does not change as seen from the explicit form (29). Note that the N_c strings stretching over the vanishing distance between the two D3-branes transform into N_c D5-branes ($1236i$), which are also wrapping the vanishing two-cycle ($6i$) between the two Kaluza–Klein (KK) monopoles. In addition to that, because of the existence of NSNS 2-form $\tilde{B}^{\mathrm{NS}}_{6i}$, there is also induced a D3-brane(123)-charge in the D5-brane in the same way as the wave in the previous section. Due to this induced charge, there is a non-trivial RR 5-form flux. We also remark that the 4D gauge coupling constant corresponds to the field $\int_{\Sigma(6i)} g_{\mathrm{s}}^{-1}\tilde{B}^{\mathrm{NS}}_{6i}\mathrm{d}x^6\mathrm{d}x^i$ and that this is not written only by $\int_{\Sigma(6i)}\mathrm{e}^{-\phi}B^{\mathrm{NS}}_{6i}\mathrm{d}x^6\mathrm{d}x^i$. By (36), we obtain the solutions for the above NSNS and RR 2-forms in the complex form

$$\begin{aligned} -\mathrm{i}\gamma &\equiv \frac{1}{2\pi\alpha'}\int_{\Sigma(6i)} g_{\mathrm{s}}^{-1}\tilde{B}^{\mathrm{NS}}_{6i}\mathrm{d}x^6\mathrm{d}x^i+\frac{\mathrm{i}}{2\pi\alpha'}\int_{\Sigma(jk)} B^{\mathrm{R}}_{jk}\mathrm{d}x^{jk} \\ &= (2N_c-N_f)\ln u+\text{const.} \end{aligned} \tag{38}$$

This is the modified twisted sector of the 2-forms on the background.

6.2 Gravity dual: suggestion

Let us discuss the supergravity dual corresponding to this configuration. For this purpose, let us reconsider the MQCD configuration first. It consists of N_f rigid D6-branes, and (NS5,D4)-branes. The state or shape of (NS5,D4)-branes is determined by the BPS condition on the N_f D6 supergravity background. The important point is that the shape of these (NS5,D4)-branes carries the information about the field theory dynamics. Therefore, in order to discuss the gravity dual, we have to extract or separate the gravity induced by these (NS5,D4)-branes from the background. This is a very difficult task. But remember that the solution of the N_f D6 SUGRA solution becomes N_f KK monopole solution in the eleven-dimensional supergravity. Moreover, after the large N_f limit, this reduces to the Z_{N_f} orbifold, that is, a locally flat metric [37, 38]. This simplifies the problem, and it will be possible to carry out the above extraction. In general, the locally flat metric transforms to another locally flat metric under the T-duality. So the above observation indicates that also in the Type IIB configuration, there is such a frame in which the background becomes locally flat.

In addition to that, we have to remember that it is only the relative (generalized) distance $\Delta\tilde{x}^6$ between the two NS5-branes that has the physical meaning of the RG-flow of the (complex) gauge coupling constant. The position itself in the direction of $\tilde{x}^6$ does not have an effect on the 4D field theory.[20] This means that there is one extra degree of freedom for the 4D field theory. This enables us to delocalize the configuration in this direction, keeping the relative distance fixed. This also simplified the problem.

As a conclusion, we can say that the problem will become easy in the following procedure.
(1) By adding the extra dimension, set the background to be locally flat.
(2) On this background, delocalize the configuration in the irrelevant direction for the 4D field theory.

But in Type IIB theory, the reliable higher-dimensional effective theory is not known. This is the point different from Type IIA theory related to the eleven-dimensional supergravity. So our following analysis is based on only the analogy of Type IIA theory, and the result is limited within the suggestion of the procedure to obtain the possible dual of the corresponding field theory.

Let us return to our model in Type IIB theory and consider the problem in the same spirit as the above. What is the appropriate parameter which should be promoted to the additional space coordinate ? The electric field (or the temporal component of the gauge field) on the D-brane will be a promising candidate. This is because this field is known to have the relation with the eleventh dimension in Type IIA, as discussed in the previous sections.

On the other hand, the coordinate x^6 in Type IIB theory does not play any active role. The configuration is delocalized in this direction and the role of the coordinate x^6 in the MQCD analysis has succeeded to the Wilson line. As a result of that, the Type IIB theory we have discussed is an almost nine-dimensional theory. So let us promote also the Wilson line to the new coordinate. Then this almost nine-dimensional theory is on the same level with the ten-dimensional Type IIA theory on the point of the degree of freedom for the space-time dimensions.

We have to note that the two-dimensional space (x^6, x^{10}) in M-theory is related by the T-duality to the torus with the complex structure $\tau \equiv C_0 + \mathrm{i}/g_\mathrm{s}$ in Type IIB theory [26]. So we can expect that the electric field and the Wilson line play the role of the new coordinates of this additional two-dimensional space for Type IIB theory.

On the KK monopole, the Wilson line and the electric field correspond to the NSNS and RR 2-forms respectively, as already seen in the sequence of the T- and S-dualities. These two parameters can be observed only on the branes in Type IIB theory. But by including them as the new space coordinates, we can formally extend our discussion to twelve-dimensional space-time. Let us define the new coordinate as

$$x^{\mathrm{NS}} \equiv \frac{a_6 g_\mathrm{s}}{2}, \qquad x^{\mathrm{R}} \equiv \frac{\chi}{2}. \tag{39}$$

Note that x^{R} and x^{NS} are the periodic coordinates. But in our discussion, the Wilson line x^{NS} always appears in the form $\mathrm{d}\tilde{x}_{\mathrm{NS}} \equiv \mathrm{e}^{-\phi}\mathrm{d}x_{\mathrm{NS}}$ with the string coupling constant $g_\mathrm{s} \to 0$. So the coordinate $\tilde{x}_{\mathrm{NS}}$ runs from $-\infty$ to $+\infty$ in the same way as a_6 or $\tilde{a}_6$. This also means that there is no S-invariance of $SL(2,Z)$ and the S-transformation is fixed[21] in our analysis.

How can we lift the ten-dimensional supergravity solution to the twelve-dimensional solutions? The hint is given by the T-invariance of $SL(2,Z)$ and the analogy of the lift from Type IIA to eleven-dimensional supergravity. We suggest the form[22]

$$\begin{aligned}
&\mathrm{d}s_{12}^2 = \mathrm{e}^{-\phi/2} g_\mathrm{s}^{1/2}\mathrm{d}s_{10}^2 + \mathrm{e}^{\phi} g_\mathrm{s}^{-1}\mathrm{d}s_2^2, \qquad (40)\\
&\mathrm{d}s_2^2 \equiv \left(\frac{g_\mathrm{s}\alpha'}{R_6}\right)^2\\
&\quad\times\Bigg[\mathrm{e}^{-2\phi}\mathrm{d}x_{\mathrm{NS}}^2\\
&\qquad + \left\{\mathrm{d}x_R - \mathrm{d}x^M \oint \left(B_{6M}^{\mathrm{R}} - C_0 B_{6M}^{\mathrm{NS}}\right)\frac{\mathrm{d}x^6}{2\pi\alpha'}\right\}^2\Bigg],
\end{aligned}$$

where M denotes the indices for the ten-dimensional space-time and runs from 0 to 9. The factor $\frac{g_\mathrm{s}\alpha'}{R_6}$ corresponds to the radius of the eleventh dimension in Type IIA theory. We can see that by the above warped factor of the dilaton, the Einstein action $\int\sqrt{-g^{(12)}}R^{(12)}$ in twelve dimensions reduces to the ten-dimensional action in the string frame $\int\sqrt{-g^{(10)}}\mathrm{e}^{-2\phi}R^{(10)}$.

Note that the degrees of freedom for the metric are the same as eleven-dimensional supergravity. This is because the relative factor for $\mathrm{d}x_{\mathrm{NS}}^2$ and $\mathrm{d}x_{\mathrm{R}}^2$ is fixed by T-invariance and there is the constraint that all the fields (and metric) are independent of x^6 with the isometry. The latter reason

[20] This direction has physical meaning for the $4+1$ dimensional field theory on the D4-branes. For the line-compactified theory ($3+1$ dimensional theory), this direction loses physical significance except for the relative compactified length. In fact, the MQCD supersymmetric cycle is determined up to the scale and phase transformation of the holomorphic coordinate y. This transformation changes the form of the Seiberg–Witten curve, but does not change the mass formula for the soliton. These degrees of freedom originate from the ones existing because we can choose the origin anywhere for (x^6, x^{10}) space

[21] This is also seen from the fact that the radius in x^6-direction is infinite in Type IIA MQCD configuration, there is no symmetry to exchange the radii for the directions of x^6 and x^{10}

[22] The factors $g_\mathrm{s}^{1/2}$ and g_s^{-1} are required for our convention in order to kill the g_s dependence in $\mathrm{e}^{-\phi/2}$ and e^{ϕ}, respectively

comes from the requirement of the T-dual of Type IIA theory.

This is the important point. There is the well-known fact that there is no supergravity theory in the full twelve-dimensional space-time. But we have to note that the above expression is defined only under the above conditions. As a result of that, the above expression is essentially an eleven-dimensional one. So the no-go theorem in the full twelve-dimensional space-time does not mean that supersymmetry does not exist in our model.

By this lift rule, the "background" (29) becomes the KK monopole solution,

$$\mathrm{d}s_{12}^2 = \eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu + \sum_{i=7,8,9}(\mathrm{d}x^i)^2 + (\mathrm{d}x^6)^2 + \mathrm{d}s_4^2,$$

$$\mathrm{d}s_4^2 \equiv (2\pi\alpha')^2\left\{H\left(|\mathrm{d}u|^2 + \Lambda^2\mathrm{d}a_6^2\right)\right. \quad (41)$$

$$\left. +H^{-1}\Lambda^2\left[\mathrm{d}\chi + N_f\left(\frac{\Lambda a_6}{R}-1\right)\mathrm{d}\varphi\right]^2\right\},$$

where we use the same notation used in (22) and (29). Next, let us take the large N_f limit along with the limit in the previous sections. By this limit, the warped factor H reduces as $H = 1 + \Lambda N_f/R \to \Lambda N_f/R$. Then we obtain the locally flat metric

$$\mathrm{d}s_{12}^2 = \eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu + \sum_{i=7,8,9}(\mathrm{d}x^i)^2 + (\mathrm{d}x^6)^2 + (2\pi\alpha')^2\left|\mathrm{d}\mathcal{M}^{(4)}_{\mathrm{Z_{N_f}}}\right|^2,$$

$$\left|\mathrm{d}\mathcal{M}^{(4)}_{\mathrm{Z_{N_f}}}\right|^2 \equiv |\mathrm{d}V_1|^2 + |\mathrm{d}V_2|^2,$$

$$V_1 \equiv (2N_f)^{1/2}\Lambda\left(a_6+\frac{R}{\Lambda}\right)^{1/2}\mathrm{e}^{\mathrm{i}\chi/2N_f},$$

$$V_2 \equiv (2N_f)^{1/2}u\left(a_6+\frac{R}{\Lambda}\right)^{-1/2}\mathrm{e}^{-\mathrm{i}\chi/2N_f}. \quad (42)$$

Note that in our notation χ has the period 4π and this leads to the Z_{N_f} orbifold identification $(V_1, V_2) \sim (\mathrm{e}^{2\pi\mathrm{i}/N_f}V_1, \mathrm{e}^{-2\pi\mathrm{i}/N_f}V_2)$. We can see that the above complex coordinates $\{V_1, V_2\}$ have relations with the holomorphic coordinates $\{y, w\}$ of the Taub-NUT space as follows:

$$y \equiv \mathrm{e}^{(a+\mathrm{i}\chi)/2}\left(a_6+\frac{R}{\Lambda}\right)^{N_f/2} \overset{N_f\to\infty}{\sim} (2N_f)^{-N_f/2}\left(\frac{V_1}{\Lambda}\right)^{N_f},$$

$$w \equiv \mathrm{e}^{-(a+\mathrm{i}\chi)/2}\left(\frac{u}{\Lambda}\right)^{N_f}\left(a_6+\frac{R}{\Lambda}\right)^{-N_f/2} \overset{N_f\to\infty}{\sim} (2N_f)^{-N_f/2}\left(\frac{V_2}{\Lambda}\right)^{N_f}.$$

Then, our problem reduces to the embedding of the three kinds of "matter" although they are originally undivided.
(1) The two KK monopoles;
(2) the complex 2-form (38) which exists between them;
(3) the D3-brane charge induced in the N_c external D5-branes

Let us consider the contribution coming from each part and discuss how to construct the gravity dual. First, let us concentrate on the two KK monopoles. Note that in Sect. 4, we can see that the sum for $\tilde{a}_6 + \mathrm{i}\chi$ on each D3-brane is also non-trivial. We can see from (32) and (33) that

$$\ln y_{\mathrm{S}} \equiv \ln y_+ + \ln y_- = \frac{1}{2}\left\{\tilde{a}_{6+} + \tilde{a}_{6-} + \mathrm{i}\left(\chi_+ + \chi_-\right)\right\} = N_f\ln\left(\frac{u}{\Lambda}\right) + \text{const.} \quad (43)$$

After the large N_f limit, we obtain the corresponding position of the whole of the two D3-branes as $N_f \ln V_2^{\mathrm{S}}=$ constant. In the same way, after the sequence of the T- and S-dualities in Sect. 6.1, we can reach the same conclusion – the whole of the two KK monopoles are located at $N_f \ln V_2 = N_f \ln V_2^{\mathrm{S}}$: constant. Of course, there remains the relative distance which corresponds to the complex 2-form (38). Let us leave the contribution from this 2-form for the next discussion and concentrate on the contribution from the two KK monopoles themselves.

Note that we cannot distinguish the direction of the KK monopole world-volume from the direction in which the KK monopole charge is delocalized or distributed. For example, there are two kinds of Type IIA KK monopoles from the point of view of M-theory. One has the world-volume in the direction of the eleventh dimension and the other is delocalized in this direction. The former type is obtained by the dimensional reduction from the KK monopole in M-theory with respect to the eleventh dimension. We can obtain the latter type by taking the T-dualities from the Type IIA NS5-brane, for example, T_{56}-dualities from the Type IIA NS5-brane(12345). But both types of Type IIA KK monopole are described by the same classical solution in M-theory.

This means that, when we delocalize the whole of the two KK monopoles in the direction of $N_f \ln V_2$, we can obtain the ordinary Type IIB KK monopole solution which is non-trivial only in the directions $\{x^6, x^7, x^8, x^9\}$ and has the KK monopole charge with respect to the compactified direction of x^6. In the region where $x^i \sim 0$ $(i = 7, 8, 9)$, it is known that the supergravity solution for the two overlapping KK monopoles reduce to the orbifold $\mathrm{R}^4/\mathrm{Z}_2$ [37].[23]

Note that on the space of $V_2 = V_2^{\mathrm{S}}$: constant, V_1 is the same as u from the definition (42). So we can interpret u in (38) as V_1 on the plane, $V_2 = V_2^{\mathrm{S}}$: constant.

Therefore, our problem reduces to the embedding of the remaining two kinds of "matter" into the locally flat

[23] This is the same procedure as we have followed for the N_f KK monopole solution

background such that

$$\mathrm{d}s_{12}^2 = \left\{\eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu + (2\pi\alpha')^2\left|\mathrm{d}\mathcal{M}_{\mathrm{Z}_2}^{(4)}\right|^2 + (2\pi\alpha')^2|\mathrm{d}V_1|^2\right\} + (2\pi\alpha')^2|\mathrm{d}V_2|^2\,, \tag{44}$$

where we use $\left|\mathrm{d}\mathcal{M}_{\mathrm{Z}_2}^{(4)}\right|^2$ as the symbol which expresses the 4D locally flat space with the Z_2 orbifold identification. The two kinds of "matter" are given by
(1) the twisted sector on the $\mathrm{R}^4/\mathrm{Z}_2$ orbifold fixed point

$$-\mathrm{i}\gamma = N_f\left(\ln V_2^- - \ln V_2^+\right) = (2N_c - N_f)\ln V_1 + \text{const;} \tag{45}$$

(2) the D3-brane charge induced in the N_c external D5-branes wrapping the vanishing two cycle on the orbifold.

Note that all the fields become independent of V_2 after being delocalized in this direction. As a result of that, this extra two-dimensional space does not play an important role for the remaining ten-dimensional theory. So we can conclude that what we have done is adding the extra two-dimensional space (39) and to pick up the unimportant other two-dimensional space (V_2-space) from the twelve-dimensional space-time. This is a procedure similar to the M-theory flip. In this remaining ten-dimensional space-time, the generalized twisted sector becomes the ordinary one.

On the other hand, in F-theory it is known that the extra two-dimensional space corresponds to the space for the dilaton and axion of Type IIB theory. In the context of F-theory, our procedure is the replacement of the two-dimensional space for the non-trivial dilaton and axion with another two-dimensional space for the constant dilaton and axion. That is, we take the frame of the (remaining) ten-dimensional case in which the dilaton and axion are constant. Our suggestion is that this remaining ten-dimensional space-time would be the dual of the corresponding field theory.

We also have to comment on the fundamental region of the 4D Z_{N_f} orbifolded space. The fundamental region can be taken as $\mathbf{C}\times\mathbf{C}/Z_{N_f}$. When we delocalize the configuration in the V_2-space, the region of this space is $\mathbf{C}/Z_{N_f}$ because it is in the form of $N_f\ln V_2$ coming from (43) that we delocalize the configuration.[24] As a result, the V_1-space spans the whole complex plane. In other words, the Z_{N_f} orbifold identification is invisible for the remaining ten-dimensional space-time.

Then we can see that the above configuration is the same as that of pure SYM theory except the values of the D5- and D3-charge. This is consistent with the fact that at one-loop level, the structure of the pure SYM vacua is qualitatively the same as the Coulomb branch of SQCD.

The ten-dimensional solution can be obtained by modifying the result for pure SYM [3]. The authors of this reference have discussed the supergravity solution for the N D5-branes wrapping on the vanishing two-cycle on the fixed point of the $\mathrm{R}^4/\mathrm{Z}_2$ orbifold. But with only a bit of change about the D-brane charge, we can formally generalize their result.

The result is summarized as follows:

$$\mathrm{d}s_{12}^2 = \mathrm{d}s_{10}^2 + (2\pi\alpha')^2|\mathrm{d}V_2|^2 \tag{46}$$

$$(2\pi\alpha')^{-1}\mathrm{d}s_{10}^2 = \frac{\rho^2}{f(|V_1|,v)^{1/2}}\{\eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu\} + \frac{f(|V_1|,v)^{1/2}}{\rho^2}\left\{|\mathrm{d}V_1|^2 + \mathrm{d}\mathcal{M}_{\mathrm{Z}_2}^2\right\},$$

$$f(|V_1|,v) = 8\pi g_{\mathrm{s}}\mathrm{Q}_{\mathrm{D3}} + 2\left(2\pi g_{\mathrm{s}}\mathrm{Q}_{\mathrm{D5}}\right)^2\left[\ln\left\{\frac{\rho^4}{v^2}\right\} + \frac{|V_1|^2}{v^2} + \text{const.}\right],$$

$$\rho^2 \equiv |V_1|^2 + v^2, \qquad v \equiv \frac{R_6^{1/2}}{\alpha'\pi}\left(\sum_{i=7,8,9} {x^i}^2\right)^{1/4},$$

$$C_{(4)} = \rho^4 f(|V_1|,v)^{-1}\mathrm{d}x^0\wedge\mathrm{d}x^1\wedge\mathrm{d}x^2\wedge\mathrm{d}x^3,$$

$$B_{(2)}^{\mathrm{R}} = b^{\mathrm{R}}\omega_{(2)}, \qquad \tilde{B}_{(2)}^{\mathrm{NS}} = \tilde{b}^{\mathrm{NS}}\omega_{(2)},$$

$$\gamma \equiv \tau\tilde{b}^{\mathrm{NS}} - b^{\mathrm{R}} = \mathrm{i}Q_{\mathrm{D5}}\ln V_1 + \text{const.}$$

$$\tau = \frac{\mathrm{i}}{g_{\mathrm{s}}} + C_0 \;:\; \text{const.},$$

where we denote the D3- and D5-charges by Q_{D3} and Q_{D5}. In these authors' case of pure $SU(N_c)$ SYM, the D5-charge is $Q_{\mathrm{D5}} = 2N_c$, and for the D3-charge they have suggested $Q_{\mathrm{D3}} = N_c/2$. We have replaced the original coordinate u by V_1 as explained. In the above equation, we denote the 2-form which is dual to the vanishing two-cycle Σ as $\omega_{(2)}$. We normalize this integral of the 2-form over Σ by $\frac{1}{2\pi\alpha'}\int_\Sigma\omega_{(2)} = 1$. This 2-form also satisfies the anti-self-duality condition, $\omega_{(2)} = -*\omega_{(2)}$. The components of the above NSNS and RR 2-forms $\left\{\tilde{B}_{(2)}^{\mathrm{NS}}, B_{(2)}^{\mathrm{R}}\right\}$ are essentially the same as $\tilde{B}_{6i}^{\mathrm{NS}}$ and B_{jk}^{R}, that we have obtained by T- and S-dualities in (37). In the following discussion, we set the RR 0-form C_0 to zero.

Then let us consider the supergravity solution for our configuration. It is easy to see that in our case, the D5-charge is $Q_{\mathrm{D5}} = 2N_c - N_f$. What about Q_{D3}? As we have commented before, this charge is determined by the NSNS 2-form field $\tilde{B}_{6i}^{\mathrm{NS}}$ on the N_c D5-branes. We approximate this as

$$Q_{\mathrm{D3}} = \frac{N_c}{4\pi^2\alpha'}\int_\Sigma \tilde{B}_{(2)}^{\mathrm{NS}}\Big|_{V_1=\epsilon} = \frac{N_c}{2\pi}\,\tilde{b}^{\mathrm{NS}}\Big|_{V_1=\epsilon}, \tag{47}$$

where ϵ means the low energy cut-off of V_1 in order to avoid the region where $\tilde{b}^{\mathrm{NS}}$ vanishes. Note that our approximations in Sect. 3 and 4 of the bifundamental matters are broken down in this region. This is because they are not heavy any more and become massless. This is the typical

[24] We have to emphasize that it is keeping V_1 fixed when we delocalize the configuration in the V_2 space. This requires another (discrete) phase transformation for V_1, according to the N_f regions of the V_2 space. This kills the phase transformation of V_1 by the original $\mathbf{R}^4/Z_{N_f}$ identification

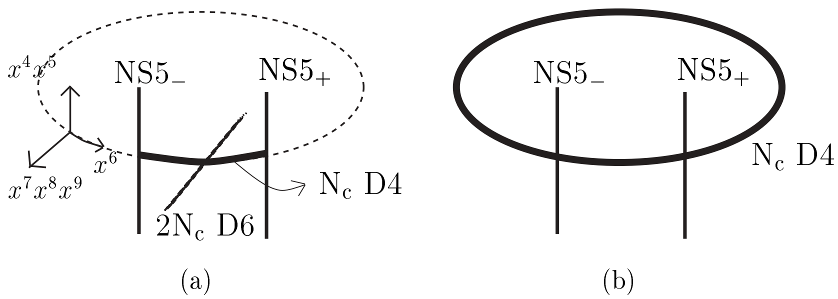

Fig. 4. Two conformal configurations: **a** partially wrapping N_c D4-branes **b** completely wrapping N_c D4-branes

limit for the perturbative analysis. The non-perturbative effects will cure this kind of singularity. Then we can also expect that the above D3-charge is determined by the low energy effective coupling constants for the $U(1)^{N_c-1}$ gauge theories coming from the broken $SU(N_c)$ gauge symmetry. We need the Seiberg–Witten curve to determine these coupling constants, but this is beyond our current analysis.

We also have to comment on the scale of the Higgs branch. In the directions of $\{x^7, x^8, x^9\}$, we need the same limit as that of the MQCD analysis:

$$x^i \to 0 \qquad \tilde{v}^2 \equiv \frac{1}{\alpha'^{3/2}} \left(\sum_{i=7,8,9} {x^i}^2 \right)^{1/2}, \quad \text{fixed.} \tag{48}$$

This scale $\tilde{v}$ corresponds to the directions of the vacuum expectation values for the quarks in the fundamental representations and does not depend on the string coupling constant. In fact, the Higgs branch is known to have no quantum correction. Compared to v, we can easily see that $\tilde{v} \gg v \sim 0$ in our model (see (28) and (48)). Then only a V_1 dependence remains in (46) because the $\tilde{v}$ dependence never appears in this solution. Especially ρ is determined by only V_1 and the holographic energy scale is expected to be V_1. These facts show that the above result corresponds to the Coulomb branch, not the Higgs branch.

Note that the complex field $\gamma \equiv \tau \tilde{b}^{\mathrm{NS}} - b^{\mathrm{R}}$ corresponds to the complex gauge coupling constant of the gauge theory. This theory is realized on the N_c D5-branes wrapping on the vanishing two-cycle. In our analysis, the NSNS 2-form is generalized as compared to the ordinary one on the flat background (pure SYM), but reproduces the correct behavior of the gauge coupling constant for SQCD.

In [16,17], it is suggested that this typical ratio of 1/2 between N_c and N_f originates from the constant $b^{\mathrm{NS}}/2\pi = 1/2$ on the orbifold.[25] This is an interesting suggestion, but it seems to be different from our result about the RG-flow. Our result is independent of this value. Moreover, in their model, this typical value of the NSNS 2-form field induces the D5-brane charge in the world-volume of the N_f D7-branes. In our model, the induced D5-charge is expected to come from the two Kaluza–Klein monopoles. These differences might be explained in terms of the Type IIA counterpart of the configuration; their configuration is the one in which D6-branes would be dynamical as D4- and NS5-branes, rather than being the background.

It is important to comment on the case of $N_f = 2N_c$. In this case, the D5-charge vanishes, and $\tilde{b}^{\mathrm{NS}}$ is the constant, which leads to the gauge coupling constant determined by the constant $\tilde{b}^{\mathrm{NS}}/g_{\mathrm{s}}$. The solution reduces to

$$\mathrm{d}s_{12}^2 = \mathrm{d}s_{10}^2 + (2\pi\alpha')^2 \left|\mathrm{d}V_2\right|^2 ,$$

$$(2\pi\alpha')^{-1}\, \mathrm{d}s_{10}^2 = \frac{\rho^2}{(4 g_{\mathrm{s}} N_c \tilde{b}^{\mathrm{NS}})^{1/2}} \left\{ \eta_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu \right\} + \frac{(4 g_{\mathrm{s}} N_c \tilde{b}^{\mathrm{NS}})^{1/2}}{\rho^2} \left\{ \left|\mathrm{d}V_1\right|^2 + \mathrm{d}\mathcal{M}^2_{\mathrm{R}^4/\mathrm{Z}_2} \right\} ,$$

$$C_{(4)} = \frac{\rho^4}{4 g_{\mathrm{s}} N_c \tilde{b}^{\mathrm{NS}}} \mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 .$$

We have to be careful with the region where the description of the supergravity will be correct. We keep the ratio $1/g_{\mathrm{YM}}^2 = \tilde{b}^{\mathrm{NS}}/g_{\mathrm{s}}$ fixed with $g_{\mathrm{s}} \to 0$ and $\tilde{b}^{\mathrm{NS}} \to 0$. In addition to that, we also have to consider the region

$$g_{\mathrm{s}} \mathrm{Q}_{\mathrm{D3}} = \frac{g_{\mathrm{s}} N_c}{2\pi} \tilde{b}^{\mathrm{NS}} \gg 1 . \tag{49}$$

This means that we need the large N_c limit in the same way as the other known SUGRA solutions.[26]

Note that except for $\rho \sim |V_1|$ in our case, the above solution is similar to that of [23]. But our expression of the D3-brane charge has a $\tilde{b}^{\mathrm{NS}}$ dependence, but the one of these authors does not. The physical meaning of this difference can be explained as follows. Their configuration consists of N_c D5-branes with $(N_c b^{\mathrm{NS}})/2\pi$ D3-brane charge and N_c anti-D5-branes with $\left(2\pi - b^{\mathrm{NS}}\right) N_c/2\pi$ D3-brane charge. As a result of that, the total D3-brane charge is N_c with vanishing D5-charge. The dependence on b^{NS} is gone. We can understand this difference more clearly in the MQCD configuration as depicted in Fig. 4[27] In their case, the N_c

[25] This is based on the result that the constant $b^{\mathrm{NS}}/2\pi$ on the orbifold is obtained as 1/2 when the perturbative string sigma model is used [39]. But as discussed in [9], in general, we can have an arbitrary value in the region $2\pi > b^{\mathrm{NS}} \geq 0$. This is also seen in the fact that this parameter corresponds to the arbitrary distance between the two NS5-branes in Type IIA theory

[26] We also have to keep the ratio N_f/N_c fixed

[27] For the comparison of the two cases, Fig. 4 is depicted by the same coordinates in the region $|u| \gg N_f \Lambda = 2N_c \Lambda$,

D4-branes are wrapping on the circle completely, but in our case they are wrapping on only a part of the circle.

7 Speculations on the non-perturbative effect

In the previous sections, we have studied the region where we can ignore the strong coupling effects – non-perturbative effects. Let us consider what will happen beyond this perturbative region. Of course, we cannot extend our analysis to this region, so we have to limit our discussion within the realm of speculation, but this kind of speculation will be useful.

For example, let us remember MQCD as suggested by [18]. This is well known to be the most successful example in taking in account the non-perturbative effects. The success of MQCD is based on the fact that in this model the D0-brane is responsible for the non-perturbative effect (instanton effect) of the 4D $\mathcal{N}=2$ SQCD. So lifting the whole system to eleven-dimensional supergravity shows the way to take in account this effect. Note that in the system in which the D0-brane does not play this key role, lifting to the 11D SUGRA does not solve the problem automatically.[28]

Imagine that we did not know the fact that 11D SUGRA includes all the effects of the D0-brane in Type IIA theory. As long as we know that the D0-brane is responsible for the 4D instanton effect, we could say at least the following: if all the effects of the D0-brane are included, the D4-brane and NS5-brane would become the same thing. This expectation comes from the property of the 4D field theory that the non-perturbative effect makes the gauge coupling constant invisible after the dimensional transmutation. Moreover, from the knowledge of the purely field theoretical analysis, we can tell what this configuration would be like – the configuration described by the Seiberg–Witten curve.

As seen in this case, the knowledge about well-known results of field theory may enable us to give some clues about the unknown aspects of string theory.

So let us speculate what will happen in the model that we have discussed. What is responsible for the non-perturbative effect in Type IIB theory? By the T-duality of the D0-branes in the MQCD configuration, we can easily find out that it is the D1-branes that play that role. These D1-branes are wrapping on the vanishing two-cycle on the orbifold. This is also confirmed by the analysis of the action [40] as done in [41, 42] for the MQCD configuration. So it is plausible that the non-perturbative effect would be included if we could add all the D1-brane effects to the previous result. But it is technically very difficult to carry out such a task directly. So we have to limit our discussion within qualitative speculations about the configuration which would be described by the Seiberg–Witten curve.[29] But this can be done without using the explicit (direct) calculations of the D1-brane effects.

First, let us consider simple pure SYM theory. In the weak coupling region, this is the Type IIB configuration discussed in [2, 3], which is also the case of $N_f=0$ in our model. In this region, the flow of the gauge coupling constant is described as the complex field $\gamma \equiv \tau b^{\mathrm{NS}} - b^{\mathrm{R}}$.[30]

From the success of MQCD, we know how this complex field behaves. Because this field corresponds to the distance between the two NS5-branes on the two-dimensional space (x^6, x^{10}) in Type IIA theory, we can get the exact behavior of this complex field from the Seiberg–Witten curve as

$$\gamma(u) = \mathrm{i}(\ln y_+ - \ln y_-),$$
$$y_\pm \equiv \sum_{n=0}^{N_c} s_n u^n \pm \sqrt{\left(\sum s_n u^n\right)^2 - \Lambda_{\mathrm{QCD}}^{2N_c}}\ . \quad (50)$$

Here $\{s_n\}$ are the moduli parameters which satisfy the conditions $s_{N_c}=1$ and $s_{N_c-1}=0$. We also denote the dynamical scale for this gauge theory as Λ_{QCD}. The above $y_\pm$ are the solution of the quadratic equation (Seiberg–Witten curve), $y + \Lambda_{\mathrm{QCD}}^{2N_c}/y = 2\sum_{n=0}^{N_c} s_n u^n$.[31]

Note that the real and imaginary parts of $\gamma(u)$ have the origin of RR and NSNS 2-form gauge field respectively, as seen in the previous sections. But we have only D5-branes and no NS5-branes in our model. So it is plausible that even in the strong coupling region we will obtain a real integer corresponding to the quantized D5-charge and no NS5-charge.

To find out what happens in the strong coupling region, let us study the complex field strength

$$\partial_u \gamma(u) = 2\mathrm{i}\left(\sum n s_n u^{n-1}\right)\left\{\left(\sum s_n u^n\right)^2 - \Lambda_{\mathrm{QCD}}^{2N_c}\right\}^{-1/2}. \quad (51)$$

As seen in the form of this field strength, there are branch cuts between the two points, say, $u_\pm^{(i)}$ $(i=1,2,\ldots,N_c)$ which satisfy $\sum s_n (u_\pm^{(i)})^n = \pm\Lambda_{\mathrm{QCD}}^{N_c}$. They reduce to $u_+^{(i)} = u_-^{(i)}$ under the condition of $\Lambda_{\mathrm{QCD}}=0$. When we integrate the field strength (51) around the pairs of these points, we

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although the AdS/CFT correspondence is not applicable in this region, but the perturbative analysis of the field theory is a good description

[28] For example, let us consider the NS5(12345)–D2(16)–NS5(12345) system which is the T-dual (T_{23}) of the MQCD configuration {NS5(12345)–D4(1236)–NS5(12345)}. The non-perturbative effect of this system is due to the D2-brane, not the D0-brane. This makes it impossible to take in account the whole non-perturbative effect only by lifting to 11D SUGRA. That is, when we lift this system to 11D SUGRA, we can distinguish the M2-branes from the two M5-brane on which the M2-branes are ending. This means that we can see the magnitude for the gauge coupling constant of 2D SYM on the D2-branes and that the dimensional transmutation does not happen yet

[29] Here we limit our discussion within the study of the configuration and flux, not the gravity solution. In [43], the gravity dual for 3D SYM was discussed in which the singularity is removed by adding the "non-perturbative gauge fields"

[30] Here we consider the general cases with $C_0 \neq 0$

[31] In MQCD, the complex coordinate y corresponds to the real coordinate (x^6, x^{10}) by the relation $\ln y = (x^6 + \mathrm{i}x^{10})/R_{10} = (\phi + \mathrm{i}\chi)/2$ in our notation

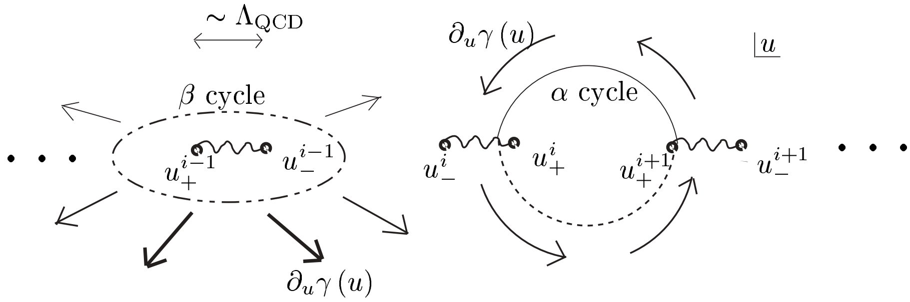

Fig. 5. The two kinds of cycles and the behavior of the field strength $\partial_u \gamma(u)$

obtain the expected result – no NS5-charge and the quantized D5-charge $\frac{1}{2\pi}\oint \partial_u\gamma = 2m$. Here m is the number of the pairs of branch points $(u_-^{(i)}, u_+^{(i)})$ surrounded by the path of the integral. This reproduces the classical picture that the D5-branes are located at the points which satisfy $\sum s_n u^n = 0$. So we can conclude that the non-perturbative effect in Type IIB theory causes the splits of these classical N_c positions of the D5-branes. This is a well-known phenomenon in 4D $N=2$ gauge theories. This has a lot of implications. It shows that we cannot exactly tell where the D5-branes are located. They seem to spread on the u-plane and make up the different types of singularity from the point-like source – a branch cut. This corresponds to the situation in MQCD that the D4-branes become indistinctive of the NS5-branes after the strong coupling effect. On the field theory side, this is the manifestation of the dimensional transformation.

In addition to the above type of integral path (a β-cycle), there is also another type of path of the integral, called an α-cycle. This is the path which runs around the points, say $(u_+^{(i)}, u_+^{(i+1)})$ crossing the ith and $(i+1)$th branch cuts (see Fig. 5). By this α-cycle integral for the field strength (51), we can easily see $\frac{1}{2\pi}\oint_\alpha \partial_u\gamma = 0$. This means that there is no source from which the flux goes out. It leads to the fluxes going around the α-cycles from one branch cut to another branch cut.

Can we give geometrical meaning to this $\gamma(u)$? We can interpret this complex function to express the point $(b^{\rm NS}, -b^{\rm R})$ on the torus with constant complex structure $\tau = \mathrm{i}/g_{\rm s} + C_0$. In our analysis, we have to limit ourselves within the region $g_{\rm s} \ll 1$ and $b^{\rm NS} \ll 1$ with an arbitrary magnitude of the ratio $b^{\rm NS}/g_{\rm s}$. As a result, one of the two periods of this torus is finite and the other is infinite. This leads to the conclusion that this complex function, $\gamma(u)$, expresses the arbitrary point in the belt-like two-dimensional plane with the topology $\mathbf{R}\times\mathbf{S^1}$. This also means that we do not have complete invariance under the $SL(2,Z)$ transformation – there is the invariance under a T-transformation which originates from the periodicity of the x_{10}-direction in M-theory, but no invariance under S-transformation.[32] Of course, we can directly see this fact from the form of this complex function.

We comment here on the Seiberg–Witten 1-form. This is written $\lambda_{\rm SW} \equiv u\mathrm{d}\gamma$ and gives us the exact expression for the effective gauge coupling constants of the low energy $U(1)^{N_c-1}$ gauge theory of the 4D $\mathcal{N}=2$ $SU(N_c)$ gauge theory. The $U(1)$ effective gauge coupling constant (perturbatively) corresponds to the value of the field $g_{\rm s}^{-1}b^{\rm NS}$ at the point where each D5-brane is located. As seen in our discussion, this also gives the expression for the D3-charge induced in each D5-brane. So we can expect that the exact result for the effective coupling constant will also give us the exact expression for the D3-brane charge. This is also the same as the case with fundamental matter particles if we replace $b^{\rm NS}$ with $\tilde{b}^{\rm NS}$. The calculation of these effective gauge coupling constants has been done many times, so we do not repeat this analysis here. We limit our discussion to a comment on this.

In summary, the non-perturbative (D1-brane) effect will be speculated on below.
(1) The classical δ function-like singularities as the source of the D5-charge change into those of the branch cuts.
(2) There is the new type of "flux"[33] which goes round between one branch cut and another.

Next, let us consider the case including the (massless) N_f fundamental matter particles.[34] This is almost the same as the pure Yang–Mills case except that the Seiberg–Witten curve is different. This difference leads to the modification of γ:

$$\gamma(u) = \mathrm{i}(\ln y_+ - \ln y_-), \tag{52}$$
$$y_\pm \equiv \sum_{n=0}^{N_c} s_n u^n \pm \sqrt{\left(\sum s_n u^n\right)^2 - \Lambda_{\rm QCD}^{2N_c-N_f} u^{N_f}} \ .$$

[32] This fact is also easily confirmed by the observation of the following: in the corresponding MQCD configuration, we have to set the radius of the x^6-direction infinite in order to avoid the NS5-branes crossing each other. An exception with S-invariance is the case with conformal invariance known as the elliptic model in which the NS5-branes are straight without crossing each other

[33] The quotation marks are added to mean that this is the flux after taking in account non-perturbative D-string effects

[34] We limit our analysis in the region $N_f < 2N_c$ in which the gauge theory is asymptotically free

In the above, the $y_{\pm}$ are the solutions of the quadratic equation (Seiberg–Witten curve), $y + \Lambda_{\mathrm{QCD}}^{2N_c-N_f} u^{N_f}/y = 2\sum_{n=0}^{N_c} s_n u^n$. Note that y is not the same coordinate as that of pure SYM theory, but the same as the one that appeared in our analysis of the previous sections.[35]

Let us rewrite the expression for γ as

$$\begin{aligned}\gamma(u) &= \mathrm{i}\left\{2\ln\left(\frac{\sum_{n=0}^{N_c} s_n u^n + \sqrt{\left(\sum s_n u^n\right)^2 - \Lambda_{\mathrm{QCD}}^{2N_c-N_f} u^{N_f}}}{\Lambda_{\mathrm{QCD}}^{N_c}}\right)\right. \\ &\left. - N_f \ln\left(\frac{u}{\Lambda_{\mathrm{QCD}}}\right)\right\}. \end{aligned} \tag{53}$$

It is easy to see from the first term that there are N_c singularities of the branch cuts.[36] Roughly speaking, this shows that the classical δ function-like singularities of the external N_c D5-brane source change into those of the branch cuts. The second statement in pure SYM theory about the two kinds of flux is also applicable to this case with matter particles. But we have to be careful with the second term in the above expression for $\gamma(u)$. This gives an additional contribution of $-N_f$ D5-charge to the contour integral around the origin. So we can roughly say that this term is the contribution of the matter or the background, as compared with the first term. In fact, in the region $|u| \gg \Lambda_{\mathrm{QCD}}$, we can see the behavior of $\gamma(u)$ with the vanishing moduli $\{s_n\} = 0$,

$$\gamma(u) \sim \mathrm{i}\,(2N_c - N_f)\ln\left(\frac{u}{\Lambda_{\mathrm{QCD}}}\right), \tag{54}$$

where the first term in the above comes from the first term of (53). This is the perturbative RG-flow in the ultraviolet region in the 4D field theory.

Note that we can also obtain this result in the gentler region $|u|/\Lambda_{\mathrm{QCD}} > 1$ by the large N_c and N_f limit. This is the RG-flow in the region where the AdS/CFT correspondence is effective as discussed in Sect. 6.

Therefore as long as one of the above conditions is satisfied, our result for the complex field in the previous sections is trustworthy.[37]

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[35] The definition of y is given in (32) and the relation with a and the NSNS 2-form field is given in (35)

[36] There are a multiple N_f of branch points, but we can resolve this singularity by giving the mass term

[37] Our approximation about the source as the heavy bifundamental quark is justified in this region

Appendix

In this appendix, we will show that our analysis in Sects. 3.1 and 3.2 is the same when we start from the Born–Infeld action.

After taking the static gauge $\sigma_0 = t$ and $\{\sigma_4, \sigma_5\} = \{x_4, x_5\}$, and assuming that only $X^6 = X^6(x_4, x_5)$ and $A_0 = A_0(x_4, x_5)$ are the non-trivial fields, let us take the limit (3). Then the action (8) reduces to

$$\begin{aligned} S_{\mathrm{D2}}^{\mathrm{BI}} &= -\frac{1}{4\pi\Lambda}\int \mathrm{d}t\mathrm{d}u^4\mathrm{d}u^5 H\left\{G\left(\nabla_u\phi, \nabla_u a\right)\right\}^{1/2} \\ &+ \frac{\Lambda^2}{4\pi}\int \mathrm{d}t\mathrm{d}u^4\mathrm{d}u^5 \frac{N_f}{R^2}\left(\frac{\phi}{R} - \nabla_u\phi\cdot\frac{\mathbf{u}}{R}\right)a, \\ G &\equiv G\left(\nabla_u\phi, \nabla_u a\right) \\ &\equiv 1 + \Lambda^2\left\{\left|\nabla_u\phi\right|^2 - \left|\nabla_u a\right|^2\right\} - \Lambda^4\left(\nabla_u\phi\times\nabla_u a\right)^2, \end{aligned} \tag{55}$$

where we use the same convention as in Sect. 3.1. Let us add the source with $\pm N_c$ electric charges to the above action. From the action $S_{\mathrm{D2}}^{\mathrm{BI}} + \Delta S$, we can see the constraint for a (Gauss law),

$$\begin{aligned} &\nabla_u\left(HG^{\frac{-1}{2}}\nabla_u a\right) \\ &+ \Lambda^2\nabla_u\times\left\{HG^{\frac{-1}{2}}\nabla_u\phi\left(\nabla_u a\times\nabla_u\phi\right)\right\} \\ &= \frac{N_f\Lambda}{R^3}\left(\phi - \nabla_u\phi\cdot\mathbf{u}\right) \pm 4\pi N_c\delta(u^4)\delta(u^5) \\ &= \nabla_u\left(-\frac{N_f\Lambda\phi}{R}\frac{\mathbf{u}}{|u|^2} + \left\{\mathrm{sign}(\phi_0)N_f \pm 2N_c\right\}\frac{\mathbf{u}}{|u|^2}\right), \end{aligned} \tag{56}$$

where the right hand is the same as that of Sect. 3.1. Next, let us consider the equation of motion. We can easily obtain

$$\begin{aligned} &\nabla_u\left(HG^{\frac{-1}{2}}\nabla_u\phi\right) \\ &- \Lambda^2\nabla_u\times\left\{HG^{\frac{-1}{2}}\nabla_u a\left(\nabla_u\phi\times\nabla_u a\right)\right\} \\ &= -\frac{N_f\Lambda}{R^3}\left(G^{\frac{-1}{2}}\phi + \nabla_u a\cdot\mathbf{u}\right) \mp 4\pi N_c\delta^2(\mathbf{u}). \end{aligned}$$

From this form, we can see that the same additional relation $\nabla a = -\nabla\phi$ which appears in Sect. 3.1. makes the above equation equivalent to the Gauss law (56). As a result, we obtain the same equation as (14) to determine the behavior of ϕ as that of Sect. 3.1.

Then let us discuss the field χ. The electric charge $\mathrm{Q_E}$ in this case is given by the integral of the left hand as

$$\begin{aligned} \mathrm{Q_E} &= \frac{1}{4\pi}\oint *\left\{HG^{\frac{-1}{2}}\nabla_u a + \Lambda^2 HG^{\frac{-1}{2}}\tilde{\nabla}_u\phi\left(\nabla\phi\times\nabla a\right)\right. \\ &\left. + N_f\left(\frac{\Lambda\phi}{R} - \mathrm{sign}(\phi_0)\right)\frac{\mathbf{u}}{|u|^2}\right\} \\ &= \pm N_c. \end{aligned}$$

By taking the current of $\phi_0 > 0$ as the standard, we define the "dual" field χ as

$$\tilde{\nabla}_u \chi \equiv H G^{\frac{-1}{2}} \nabla_u a + \Lambda^2 H G^{\frac{-1}{2}} \tilde{\nabla}_u \phi \left(\nabla\phi \times \nabla a\right) + N_f \left(\frac{\Lambda\phi}{R} - 1\right) \frac{\mathbf{u}}{|u|^2}. \quad (57)$$

By this definition, we can see that the solution for χ is the same as (22).

In summary, the final results for ϕ and χ are the same as those of Sect. 3.1 even when we start with the Born–Infeld action (55).

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